

A Mathematical Analysis of a Prey Predator and Amensal Model

Paparo A. V

Department of Mathematics, JNTU-GV, Collège of Engineering,
Vizianagaram, A P, India.
paparao.alla@gmail.com

ABSTRACT

In this paper, we explore the dynamics of the three species model with prey, predator and amensal. The first species is the predator, second species is prey which is amensal to third species (amensal). A constant time lag (τ) is induced in the amensal species. Mathematical model is delineated by the system of delay differential equations. The co-existing state is identified and studied the local dynamics at this point. For $\tau = 0$ the system is locally asymptotically stable. For $\tau > 0$, the system admits instability tendency cause the Hopf bifurcation. The sufficient condition for existence of Hopf bifurcation is derived. The analytical results are validated using numerical simulation using constant time lag (τ) as bifurcation parameter. The critical value of the system is identified using MATLAB simulation. For certain values the system exhibits unstable nature leads to the Hopf bifurcation.

Keywords: Prey, Predator, Amensal, Co-existing state, local stability, Hopf bifurcation.

AMS Classification: 34DXX

1. INTRODUCTION

Mathematical analysis for ecological and biological models is inevitable. The predictive analysis for the ecological models using mathematical tools like differential equations was initiated by Lotka [1] and Volterra [2]. Researchers face difficulty to understand the dynamics of the prey-predator models. Mathematical approach plays a significant role to interact and reveal the facts about the dynamics of the ecological phenomenon. The pioneering work of [1,2] brought the importance of developing theoretical framework in prey-predator, competitor, amensal interactions. Most researchers on the dynamics of prey-predator models are dealt global dynamics. Many parameters like fear, Allee, harvesting, delays are influence the dynamics of the model. Depending up on the interaction the system may stabilize or destabilize the system.

To comprehend more actual situations Holling type response was taken into count and the results of prey-predator dynamics was delay by [11]. Harvesting effort is one of the most influence factor in prey-predator dynamics. Different types of harvesting efforts are included in the study of prey-predation

interactions [6,16,17]. The prey harvesting using Michalis-Menten type results are more biological significant than constant and constant effort harvesting. Another important factor in stability analysis of prey-predator models is fear effect of prey species. The fear effect may stabilize or destabilize the system elaborated by [3]. Hopf bifurcation analysis is well posed in recent era, the bifurcation parameter may be taken in the model which is a significant parameter.

The system dynamics with existence of Hopf bifurcation in the prey-predator with Holling type-III functional response and predator harvesting model [4,5] draw the attention that the loss of stability by Hopf bifurcation increases the profit. So, destabilization cause the profit in revenue is a new finding when predator species is harvested.

The delays in biological systems are quite natural. These delays cause the stable equilibrium to unstable or vice versa. The switch over behavior of the system is called Hopf bifurcation. The nature of the delay argument cause unbounded growth and extinction of populations leads to instability tendency of models. These lags will change the stable equilibrium to unstable or vice-versa. In the nature most popular are models of prey-predator type models. The stability and Hopf bifurcation analysis of prey - predator models elucidate [4,12,13]. The time lags which cause the bifurcation was well posed and identified the time lags which makes the system unstable.

The study investigated [6-8] Three species models comprising prey, predator, and competitor with distributed time lags. It was proved that co-existence equilibrium point is globally asymptotically stable [6-8]. The combinations of prey-predator with sub population amensal species admit a unique co-existence point [15] and established the local and global stability at this point. The distributed time delay amensal model [9] elaborates the existence of global dynamics.

The Hopf bifurcation analysis with distributed time lag with kernel dynamics was dealt by [14] with the Hopf bifurcation parameter as kernel weight. The critical parameter value is identified using numerical simulation. The constant single delay model with prey, predator and amensal species was dealt by [10] and shown that Hopf bifurcation exist. The time lag is taken as bifurcation parameter and the system is asymptotically stable when and for $r = 0$ the system undergoes Hopf bifurcation for some critical values of $\tau > 0$.

In addition to the existing models, we proposed a model comprising predator, prey and Amensal with time delay in amensal species. The dynamics of this model and criteria for instability was explored through Hopf bifurcation analysis.

The structure of paper is outlined as follows. section 2 represents model frame work, section 3 discuss equilibrium point, section 4 covers local stability analysis, section 5 establishes the sufficient condition for existence of Hopf bifurcation and section 6 conclude with numerical simulation supporting mathematical analysis.

2. BASIC MATHEMATICAL MODEL:

Mathematical model for the proposed system with a prey predator and amensal. Here the first species is predator, second species is prey which is amensal to third species. The constant timelag (τ) is induced in amensal species. The model is explored by the following delay differential equation.

$$\begin{aligned}\frac{dx}{dt} &= a_1x \left(1 - \frac{x}{k_1}\right) + a_{12}xy \\ \frac{dy}{dt} &= a_2y \left(1 - \frac{y}{k_2}\right) - a_{21}xy \\ \frac{dz}{dt} &= a_3z \left(1 - \frac{z(1-\tau)}{k_3}\right) - a_{32}yz\end{aligned}\quad (2.1)$$

With initial condition $x(t) = x_0$, $y(t) = y_0$ & $z(t) = z_0$

Notations

x : predator population, y : prey population, z : amensal population

a_i : Growth rates of three populations, a_{12} & a_{21} prey-predator interaction rates,

a_{32} : interaction coefficient of prey and amensal species and

k_i : carrying capacities of three species.

Assumptions: (i) predator is of generalist type (ii) All interaction coefficient are taken positive which lies in between 0 & 1, growth rates and carrying capacities, population strengths are positive not necessarily lies in between 0 & 1.

Theorem 2.1: The system (2.1) admits positive solutions in R_3^+

Proof: From the system of equations (2.1) we can write,

$$\begin{aligned}x(t) &= x(0)e^{\int_0^t \{a_1(1 - \frac{x}{k_1}) + a_{12}y\} dt} > 0 \\ y(t) &= y(0)e^{\int_0^t \{a_2(1 - \frac{y}{k_2}) - a_{21}x\} dt} > 0 \\ z(t) &= z(0)e^{\int_0^t \{a_3(1 - \frac{z(1-\tau)}{k_3}) - a_{32}y\} dt} > 0\end{aligned}$$

Hence all three solutions $x(t)$, $y(t)$ & $z(t)$ are positive in R_3^+ .

3. IDENTIFY THE CO-EXISTING POINT

Derive the co-existing state yield the following result by equating the system (2.1) to zero.

$$E(\bar{x}, \bar{y}, \bar{z}) = \left(\begin{array}{c} \frac{k_1(a_1a_2 + a_{12}a_2k_2)}{(a_1a_2 + a_{12}a_{21}k_1k_2)} \\ \frac{k_2(a_1a_2 - a_{21}a_1k_1)}{(a_1a_2 + a_{12}a_{21}k_1k_2)} \\ \frac{k_3\{(a_3a_1a_2 + a_{12}a_{21}k_1k_2) - k_2a_{32}(a_1a_2 + a_{21}a_1k_1)\}}{a_3(a_1a_2 + a_{12}a_{21}k_1k_2)} \end{array} \right)$$

(3.1)

The co-existing state exist if the following conditions are satisfied.

$$i) \quad a_1 a_2 > a_{21} a_1 k_1 ii) (a_3 a_1 a_2 + a_{12} a_{21} k_1 k_2) > k_2 a_{32} (a_1 a_2 + a_{21} a_1 k_1) \quad (3.2)$$

4.LINEAR STABILITY ANALYSIS AT CO-EXISTING STATE

Theorem 4.1: The Co-existing state E $(\bar{x}, \bar{y}, \bar{z})$ is logically asymptotically stable.

Proof: The Jacobian matrix of (2.1) can be obtained as follows

$$J = \begin{bmatrix} -\frac{a_1 \bar{x}}{k_1} & a_{12} \bar{x} & 0 \\ -a_{21} \bar{y} & -\frac{a_2 \bar{y}}{k_2} & 0 \\ 0 & -a_{32} \bar{z} & -\frac{a_3 \bar{z}}{k_3} e^{-\lambda \tau} \end{bmatrix} \quad (4.1.1)$$

The characteristic equation of (4.1.1) is given by

$$\psi(\lambda, \tau) = \lambda^3 + p_1 \lambda^2 + p_2 \lambda + p_3 + e^{-\lambda \tau} (q_1 \lambda^2 + q_2 \lambda + q_3) = 0 \quad (4.1.2)$$

$$\text{Here, } p_2 = a_1 a_2 + a_1 a_3 + a_2 a_3 + \frac{4a_1 a_2 \bar{x} \bar{y}}{k_1 k_2} + \frac{4a_1 a_3 \bar{x} \bar{z}}{k_1 k_3} + \frac{4a_2 a_3 \bar{y} \bar{z}}{k_2 k_3} + \frac{2a_1 a_{32} \bar{x} \bar{y}}{k_1} + \frac{2a_2 a_{32} \bar{y} \bar{z}}{k_1} - \frac{2a_1 a_2 \bar{y}}{k_2} - \frac{2a_1 a_2 \bar{x}}{k_1} - \frac{2a_1 a_3 \bar{z}}{k_3} - \frac{2a_2 a_3 \bar{z}}{k_3} - a_{32} a_1 \bar{y} - a_{32} a_2 \bar{y} - \frac{2a_1 a_2 \bar{x}}{k_1} - \frac{2a_3 a_2 \bar{y}}{k_2},$$

$$p_3 = \frac{2a_1 a_2 a_3 \bar{y}}{k_2} + \frac{2a_1 a_2 a_3 \bar{x}}{k_1} + \frac{2a_1 a_2 a_3 \bar{z}}{k_3} + \frac{4a_1 a_2 a_{32} \bar{x} \bar{y}^2}{k_1 k_2} + \frac{8a_1 a_2 a_{32} \bar{x} \bar{y} \bar{z}}{k_1 k_2 k_3} + a_1 a_2 a_{32} \bar{y} - \frac{4a_1 a_2 a_3 \bar{y} \bar{z}}{k_2} - \frac{4a_1 a_2 a_3 \bar{x} \bar{z}}{k_3} - \frac{2a_1 a_3 a_{32} \bar{y}^2}{k_2} - \frac{2a_1 a_2 a_{32} \bar{x} \bar{y}}{k_1} - a_1 a_2 a_3 - \frac{4a_1 a_2 a_3 \bar{x} \bar{y}}{k_1 k_2}$$

$$q_2 = a_1 a_{12} \bar{x} + a_2 a_{12} \bar{y} + a_{12} a_3 \bar{y} + a_{21} a_{32} \bar{x} \bar{y} + \frac{2a_3 a_{21} \bar{x} \bar{z}}{k_3} - \frac{2a_2 a_{12} \bar{y}^2}{k_2} - a_{12} a_{32} \bar{y}^2 - a_3 a_{21} \bar{x} - a_1 a_{21} \bar{x}$$

Which can be written as $\psi(\lambda, \tau) = P(\omega) + Q(\omega)e^{-\tau \omega}$

The characteristic equation obtained from (4.1.2) by putting $\tau = 0$ given by the following equation.

$$\psi(\lambda, 0) = -\left(\frac{a_3 \bar{z}}{k_3} + \lambda\right) \left[\lambda^2 + \lambda \left(\frac{a_1 \bar{x}}{k_1} + \frac{a_2 \bar{y}}{k_2} \right) + \left(\frac{a_1 a_2 \bar{y} \bar{x}}{k_1 k_2} + a_{12} a_{21} \bar{x} \bar{y} \right) \right] = 0$$

$$\Rightarrow \left(\frac{a_3 \bar{z}}{k_3} + \lambda \right) = 0 \text{ or } \left[\lambda^2 + \lambda \left(\frac{a_1 \bar{x}}{k_1} + \frac{a_2 \bar{y}}{k_2} \right) + \left(\frac{a_1 a_2 \bar{y} \bar{x}}{k_1 k_2} + a_{12} a_{21} \bar{x} \bar{y} \right) \right] = 0$$

From the above equation $\lambda = -\frac{a_3 \bar{z}}{k_3}$ or

$$\left[\lambda^2 + \lambda \left(\frac{a_1 \bar{x}}{k_1} + \frac{a_2 \bar{y}}{k_2} \right) + \left(\frac{a_1 a_2 \bar{y} \bar{x}}{k_1 k_2} + a_{12} a_{21} \bar{x} \bar{y} \right) \right] = 0 \quad (4.1.3)$$

One of the roots is negative i.e. $-\frac{a_3 \bar{z}}{k_3}$, from the equation (4.1.3) find the remaining two roots.

The two roots have negative real roots if the trace of the equation is negative and the determinant is positive.

The trace and determinant from the equation (4.1.3) are given as follows.

$$\text{Here the trace is } -\frac{b}{a} = -\left(\frac{a_1 \bar{x}}{k_1} + \frac{a_2 \bar{y}}{k_2} \right) < 0,$$

$$\text{Determinant } \frac{c}{a} = \frac{(a_1 a_2 + a_{12} a_{21} k_1 k_2) \bar{x} \bar{y}}{k_1 k_2} > 0,$$

Hence the co-existing state is locally asymptotically stable for the system describe in equation (2.1).

Case(ii): For $\tau > 0$, the stability analysis can be studied by substituting $\tau = \pm i\omega$ in the (4.1.2) we get the following equation.

$$\begin{aligned}\psi(i\omega, \tau) &= (i\omega)^3 + p_1(i\omega)^2 + p_2(i\omega) + p_3 + e^{-i\omega\tau}(q_1(i\omega)^2 + q_2(i\omega) + q_3) = 0 \\ \psi(i\omega, \tau) &= (i\omega)^3 + p_1(i\omega)^2 + p_2(i\omega) + p_3 \\ &\quad + [\cos(\omega\tau) - i\sin(\omega\tau)(q_1(i\omega)^2 + q_2(i\omega) + q_3)] = 0 \\ \Rightarrow -\omega^2 p_1 + p_3 - q_1 \omega^2 \cos(\omega\tau) + q_3 \cos(\omega\tau) + q_2 \omega \sin(\omega\tau) + i[-\omega^3 + \omega p_2 + q_2 \omega \cos(\omega\tau) + \\ &\quad q_1 \omega^2 \sin(\omega\tau) - q_3 \sin(\omega\tau)] = 0\end{aligned}\quad (4.1.3)$$

By separating real and imaginary parts we get the following equations

$$(q_3 - q_1 \omega^2) \cos(\omega\tau) + q_2 \omega \sin(\omega\tau) = \omega^2 p_1 - p_3 \quad (4.1.4)$$

$$q_2 \omega \cos(\omega\tau) - (q_3 - q_1 \omega^2) \sin(\omega\tau) = \omega^3 - \omega p_2 \quad (4.1.5)$$

On simplification we get

$$\begin{aligned}(4.1.4)^2 + (4.1.5)^2 &\Rightarrow (q_3 - q_1 \omega^2)^2 + q_2^2 \omega^2 = (\omega^2 p_1 - p_3)^2 + (\omega^3 - \omega p_2)^2 \\ \Rightarrow \omega^6 + \omega^4(p_1^2 - 2p_2 - q_1^2) + \omega^2(p_2^2 - 2p_1 p_3 - q_2^2 + 2q_1 q_3) + q_3^2 + p_3^2 &= 0\end{aligned}$$

$$\text{Let } \psi(p) = p^3 + p^2 N_1 + p N_2 + N_3 = 0,$$

$$\text{Here } N_1 = p_1^2 - 2p_2 - q_1^2, N_2 = p_2^2 - 2p_1 p_3 - q_2^2 + 2q_1 q_3 \text{ and } N_3 = q_3^2 + p_3^2.$$

$$\therefore \psi(p) = 0$$

If we assume that $N_1 > 0$, $N_2 > 0$, $N_3 > 0$ then have no positive real roots,

$$N_1 = p_1^2 - 2p_2 - q_1^2, N_2 = p_2^2 - 2p_1 p_3 - q_2^2 + 2q_1 q_3 \text{ and } N_3 = q_3^2 + p_3^2$$

Thus if $N_1 > 0$, $N_2 > 0$, $N_3 > 0$ then there is no ω such that $i\omega$ is an Eigen value of the characteristic equation of $\psi(\lambda, \tau) = 0$

If λ will never be a purely imaginary root of equation $\psi(\lambda, \tau) = 0$, thus the real parts of all Eigen values of $\psi(\lambda, \tau) = 0$ are negative for all $\tau \geq 0$ summarizing, the above analysis, we have the following theorem.

Theorem 4.2 The system (2.1) is locally asymptotically stable at co-existing state for all τ if the following conditions hold.

$$(i) (p_1 + q_1) > 0, (p_2 + q_2) > 0, (p_3 + q_3) > 0,$$

$$(ii) N_1 > 0, N_2 > 0, N_3 > 0$$

Proof: Now if any one of N_1 , N_2 , N_3 is negative then (1) has positive root ω_0 eliminating $\cos(\omega\tau)$ from equation (4.1.4) & (4.1.5) we get

$$\begin{aligned}\cos(\omega\tau) &= \frac{\omega^2 p_1 q_3 - p_3 q_3 - \omega^4 p_1 q_1 + q_3 \omega^3 - p_2 q_3 \omega^2}{q_3^2 + q_1^2 \omega^4 - 2q_1 q_3 \omega^2 + q_2^2 \omega^2} \\ \tau_k &= \frac{1}{\omega_0} \cos^{-1} \left[\frac{\omega_0^4 (q_3 - p_1 q_1) + \omega_0^2 (p_1 q_3 + q_1 p_3 - p_2 q_3) - p_3 q_3}{q_1^2 \omega_0^4 + \omega_0^2 (q_2^2 - 2q_1 q_3) + q_3^2} \right] + \frac{2k\pi}{\omega_0}\end{aligned}\quad (4.2.1)$$

Where $k = 0, 1, 2, 3, \dots$

5. HOPF BIFURCATION:

Theorem 5.1: The sufficient condition for the system (2.1) admits bifurcation along the co-existing state E if $\tau > \tau_0$ and locally asymptotically stable If $0 < \tau < \tau_0$.

Proof: Hopf bifurcation occurs when the real part of $\lambda(t)$ become positive when $\tau > \tau_0$, and the steady state become unstable moreover, when τ passes through the critical value τ_0 .

To check this result, differentiating the equation (4.1.2) with respect to ' τ ' we get

$$3\lambda^2 \frac{d\lambda}{d\tau} + 2p_1\lambda \frac{d\lambda}{d\tau} + p_2 \frac{d\lambda}{d\tau} + e^{-\lambda\tau} \left(2q_1\lambda \frac{d\lambda}{d\tau} + q_2 \frac{d\lambda}{d\tau} \right) + (q_1\lambda^2 + q_2\lambda + q_3) \left(-\lambda - \lambda \frac{d\lambda}{d\tau} \right) e^{-\lambda\tau} = 0.$$

$$\frac{d\lambda}{d\tau} \left[3\lambda^2 + 2p_1\lambda + p_2 + e^{-\lambda\tau} (2q_1\lambda + q_2) - (q_1\lambda^2 + q_2\lambda + q_3) \lambda e^{-\lambda\tau} \right] = (q_1\lambda^2 + q_2\lambda + q_3) \lambda e^{-\lambda\tau}$$

$$\Rightarrow \left[\frac{d\lambda}{d\tau} \right]^{-1} = \frac{[3\lambda^2 + 2p_1\lambda + p_2 + e^{-\lambda\tau} (2q_1\lambda + q_2) - (q_1\lambda^2 + q_2\lambda + q_3) \lambda e^{-\lambda\tau}]}{(q_1\lambda^2 + q_2\lambda + q_3) \lambda e^{-\lambda\tau}}$$

$$\Rightarrow \left[\frac{d\lambda}{d\tau} \right]^{-1} = \frac{3\lambda^2 + 2p_1\lambda + p_2}{(q_1\lambda^2 + q_2\lambda + q_3) \lambda e^{-\lambda\tau}} + \frac{(2q_1\lambda + q_2)}{(q_1\lambda^2 + q_2\lambda + q_3) \lambda} - \frac{\tau}{\lambda}$$

$$\Rightarrow \left[\frac{d\lambda}{d\tau} \right]^{-1} = \frac{3\lambda^2 + 2p_1\lambda + p_2}{-\lambda(\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3)} + \frac{(2q_1\lambda + q_2)}{(q_1\lambda^2 + q_2\lambda + q_3) \lambda} - \frac{\tau}{\lambda}$$

Substitute $\lambda = i\omega_0$ we get

$$\left[\frac{d\lambda}{d\tau} \right]^{-1} = \frac{1}{\omega_0} \left[\frac{-3\omega_0^2 + 2ip_1\omega_0 + p_2}{(-\omega_0^3 + p_2\omega_0 + i(p_1\omega_0^2 - p_3)^2)} + \frac{(2iq_1\omega_0 + q_2)}{(-q_2\omega_0 + i(q_3 - q_1\omega_0^2))} + \tau i \right]$$

$$\left[\frac{d\lambda}{d\tau} \right]^{-1} = \frac{1}{\omega_0} \left[\frac{(-3\omega_0^2 + 2ip_1\omega_0 + p_2)(-\omega_0^3 + p_2\omega_0 - i(p_1\omega_0^2 - p_3)^2)}{(-\omega_0^3 + p_2\omega_0)^2 + (p_1\omega_0^2 - p_3)^2} + \frac{(2iq_1\omega_0 + q_2)(-q_2\omega_0 - i(q_3 - q_1\omega_0^2))}{(q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2} + \tau i \right]$$

$$\text{Real part of } \left[\frac{d\lambda}{d\tau} \right]^{-1} = \frac{1}{\omega_0} \left[\frac{(-3\omega_0^2 + p_2)(-\omega_0^3 + p_2\omega_0) + 2p_1\omega_0(p_1\omega_0^2 - p_3)^2}{(-\omega_0^3 + p_2\omega_0)^2 + (p_1\omega_0^2 - p_3)^2} + \frac{-q_2^2\omega_0 + 2q_1\omega_0(q_3 - q_1\omega_0^2)}{(q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2} \right]$$

Using the following relation

$$(-\omega_0^3 + p_2\omega_0)^2 + (p_1\omega_0^2 - p_3)^2 = (q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2,$$

$$\Rightarrow \text{Real part of } \left[\frac{d\lambda}{d\tau} \right]^{-1} = \frac{1}{\omega_0} \left[\frac{3\omega_0^5 + \omega_0^3(2p_1^2 - p_2 - 3p_2 - 2q_1^2) + (p_2^2 - 2p_1p_3 - q_2^2 + 2q_1q_3)\omega_0}{(q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2} \right]$$

$$\Rightarrow \text{Real part of } \left[\frac{d\lambda}{d\tau} \right]^{-1} = \left[\frac{3\omega_0^4 + \omega_0^2(2p_1^2 - 4p_2 - 2q_1^2) + (p_2^2 - 2p_1p_3 - q_2^2 + 2q_1q_3)}{(q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2} \right]$$

$$\Rightarrow \left[\frac{d}{d\tau} \text{Re}(\lambda) \right] = \left[\text{Re} \left[\frac{d\lambda}{d\tau} \right]^{-1} \right]_{\lambda=i\omega_0} = \left[\frac{3\omega_0^4 + \omega_0^2(2p_1^2 - 4p_2 - 2q_1^2) + (p_2^2 - 2p_1p_3 - q_2^2 + 2q_1q_3)}{(q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2} \right]$$

By using this condition $N_1 > 0$, $N_2 > 0$, $N_3 > 0$, we have $\left[\frac{d}{d\tau} \text{Re}(\lambda) \right]_{\lambda=i\omega_0} > 0$

Therefore, the Hopf bifurcation occurs at $\tau > \tau_0$.

6. NUMERICAL SIMULATION:

Hopf bifurcation analysis is illustrated for the model equations (2.1) using the following set of parameters taken (τ) as bifurcation parameter.

Notation: Fig. A & C represent Time Series plot, Fig. B & D represent Phase portrait.

Example: 6.1 let us choose the following parameters for examination $a_1=5, a_2=1, a_3=3, a_{12} = 0.5, a_{21}=0.5, a_{32}= 0.04, k_1 = 150, k_2=150, k_3 = 150, x = 3, y = 2, z = 2$.

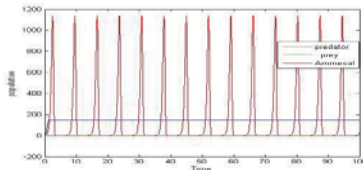


Fig. 6.1 (A)

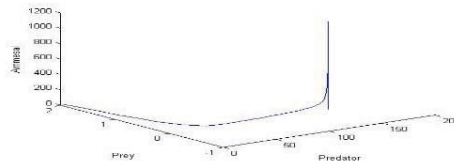


Fig. 6.1 (B)

The nature of the system (2.1) is unstable due to unbounded oscillations in prey and predator populations for $\tau = 0.6$.

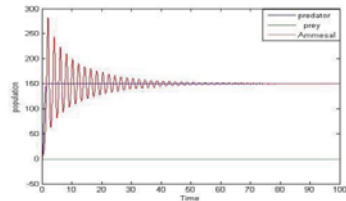


Fig. 6.1 (C)

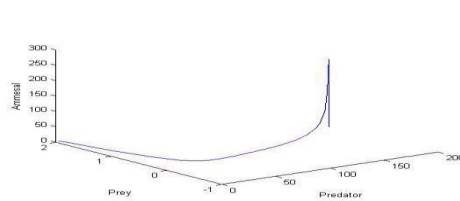


Fig. 6.1 (D)

The system (2.1) is stable for $\tau = 0.5$ as the solution curves approaching equilibrium point with diminishing amplitude.

Example: 6.2 let us choose the following parameters for examination $a_1=1, a_2=1, a_3=1, a_{12} = 0.5, a_{21}=0.5, a_{32}= 0.04, k_1 = 150, k_2=150, k_3 = 150, x = 3, y = 2, z = 2$.

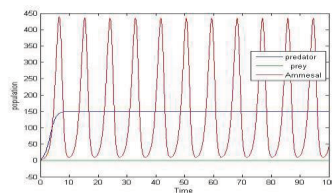


Fig. 6.2 (A)

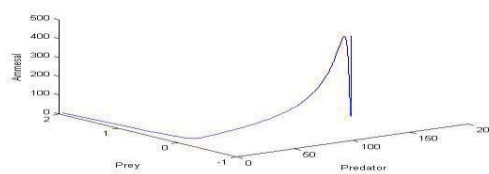


Fig. 6.2 (B)

The nature of the system (2.1) is unstable due to unbounded oscillations in prey and predator population for $\tau = 2.0$.

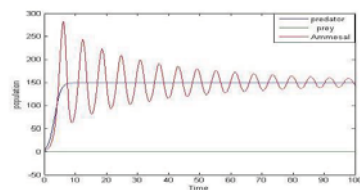


Fig. 6.2 (C)

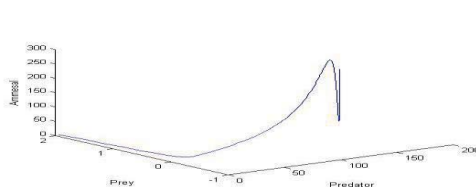


Fig. 6.2 (D)

The system (2.1) is stable due to bounded solutions, when $\tau = 1.5$.

Example: 6.3 let us choose the following parameters for examination $a_1=1$, $a_2=0.5$, $a_3=0.25$, $a_{12} = 0.5$, $a_{21}=0.5$, $a_{32}= 0.04$, $k_1 = 150$, $k_2=150$, $k_3 = 150$, $x = 3$, $y = 2$, $z=2$.

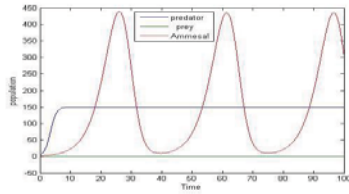


Fig. 6.3 (A)

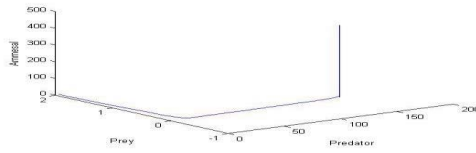


Fig. 6.3 (B)

The nature of the system (2.1) is unstable due to unbounded oscillations in prey and predator populations for $\tau = 7.0$.

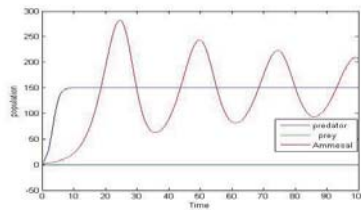


Fig. 6.3 (C)

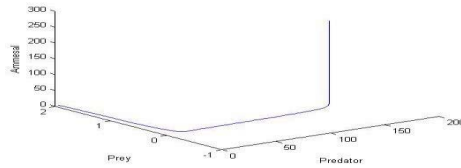


Fig. 6.3 (D)

The system (2.1) is stable due to bounded solutions, when $\tau = 6.5$.

Example: 6.4 let us choose the following parameters for examination $a_1=3$, $a_2=2.5$, $a_3=3$, $a_{12} = 0.7$,

$a_{21}=0.5$, $a_{32}= 0.04$, $k_1 = 150$, $k_2=150$, $k_3 = 150$, $x = 3$, $y = 2$, $z=2$.

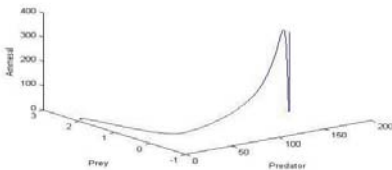
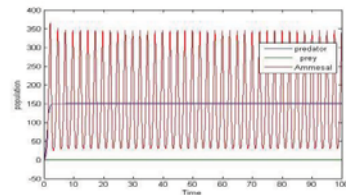


Fig. 6.4 (A)

Fig. 6.4 (B)

The unbounded periodic solutions for the system (2.1) when $\tau = 0.6$.

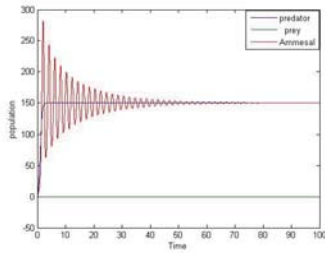


Fig. 6.4 (C)

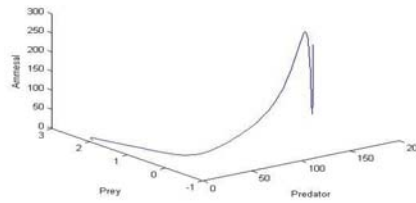


Fig. 6.4 (D)

The system (2.1) becomes stable due when $\tau = 0.5$.

7. RESULTS AND DISCUSSION:

In the proposed model involving predator, prey and amensal, a time delay is induced in the amensal species, leading to observe the dynamics of Hopf bifurcation. The system exhibits asymptotically stable for $\tau = 0$. However, for $\tau > 0$, the model exhibits instability tendency where stable equilibria become unstable. The Hopf bifurcation analysis is identified the critical value (τ) as the bifurcation parameter in the model. The system permits unstable behavior. To validate the numerical results, we derive sufficient conditions for the existence Hopf bifurcation using theorem (5.1). The numerical simulation is performed using MATLAB to determine the bifurcation parameter (τ). The critical value of τ is presented considering the four set of parametric values taken in numerical examples.

S.No.	Example	Hopf bifurcation value
1	Example 6.1	$\tau > 0.5$
2	Example 6.2	$\tau > 1.5$
3	Example 6.3	$\tau > 6.5$
4	Example 6.4	$\tau > 0.5$

Table 7.1: The critical values of the model (2.1)

The bifurcation parameter (τ) makes switch in behavior causing the stable equilibrium to point to become unstable. The extinction of prey population results in unbounded growth in amensal population leading to the occurrence of Hopf bifurcation. The predator is of generalist type and continues to sustain. These results are validated through analysis conducted using four numerical examples.

8. REFERENCES

- [1] Lotka. A.J., 1925 *Elements of physical biology*, Williams and Wilkins, Baltimore .
- [2] Volterra, 1931, *V L., econssen la theoriemathematique dela liette pou Lavie*, Gauthier- Villars, Paris.
- [3] Ruizhi Yang, Dan Jin ., 2022 , *Dynamics in a predator-prey model with memory effect in predator and fear effect in prey*. Electronic Research Archive vol 30(4): 1322-1339. doi: 10.3934/era.2022069

- [4] Nadjah.K, Salah, A.M,2021, *Stability and Hopf bifurcation of the coexistence equilibrium for differential-algebraic biological economic system with predator harvesting*. Electron. Res Arch., 29(1), 1641-1660.
- [5] Jiao Jiang, Xiu Shuai Li., Xiaotian., 2023 , *The Dynamics of Bio economic Model with Michaelis-Menten Type Prey Harvesting*, Bull.Malays.Math.Sci.Soc. 46-57
<https://doi.org/10.1007/s40840-022-01452-4>
- [6] Papa Rao A.V., Lakshmi Narayan K., 2017 *Dynamics of prey predator and competitor model with time delay*. International Journal of Ecology & Development, Vol 32, Issue No. 1, 75-86.
- [7] Papa Rao A.V, N.V.S.R.C. Murty Gamini., 2018, *Dynamical Behaviour of Prey Predators Model with Time Delay* International Journal of Mathematics And its Applications., Vol 6 issue 3 27-37.
- [8] Papa Rao A.V, N.V.S.R.C. Murty Gamini., 2019 ,*Stability Analysis of a Time Delay Three Species Ecological Model*. International Journal of Recent Technology and Engineering (IJRTE)., Vol 7 Issue-6S2, 839-845.
- [9] Papa Rao. A. V, Lakshmi Narayan. K, Kondala Rao.K.,2019, *Amensalism Model A Mathematical Study*, International Journal of Ecological Economics & Statistics (IJEES) , Vol 40, issue 3, (2019) 75-87.
- [10] Paparao A. V, G A L satyavathi, K. Sobhan Babu., 2021 *Three species Amensalism model with time delay*. International journal of tomography and simulation, 2021 Vol 34, Issue 01,66-78.
- [11] Chen, H, Zhang, C.,2022, *Analysis of the dynamics of a predator-prey model with Holling functional response*. J. Nonl. Mod. Anal. vol 4, 310–324.
- [12] Liu, W, Jiang, Y.L ., 2018 ,*Bifurcation of a delayed gauge predator-prey model with Michaelis-Menten type harvesting*. J. Theor. Biol. 438, 116–132.
- [13] Ranjith Kumar G, Kalyan Das, Lakshmi Narayan Ravindra Reddy B.,2019 *Crowding effects and depletion mechanisms for population regulation in prey-predator intra- specific competition model*. Computational Ecology & software, vol 9(1), 19-36.
- [14] Papa Rao A.V., N.V.S.R.C. Murty Gamini ., 2021 *Hopf bifurcation analysis in three species ecological model*, Southeast Asian journal of mathematics and mathematical sciences, Vol 16(1A) 177-184.
- [15] Kondala Rao K., Lakshmi Narayan K., 2016, *Stability Analysis of Three Species Syn-Ecological Model with Prey-Predation and Amensalism*, the Bulletin of Calcutta Mathematical Society, Volume-108, No 1, 63-76.
- [16] Paparao A.V, G A L satyavathi, K. Sobhan Babu., 2023, *Dynamics of Prey- Predator Model with constant effort Harvesting of Prey*. International Journal of Ecological Economics & Statistics (IJEES), vol 44 issue 01, 40-50.
- [17] Paparao A.V, G A L Satyavathi, K. Sobhan Babu. 2023, *Dynamics of Prey- Predator Model with Michaelis-Menten Type Prey Harvesting*; International Journal of Ecology & Development. vol 38 issue 02 67-77.