

Dynamical Analysis of an Exploited Prey-Predator System in Fuzzy Environment

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ABSTRACT

Understanding the uncertainty in nature is a significant factor for an ecosystem. Considering this impreciseness we present the dynamics of an exploited prey-predator model in fuzzy environment. Here the uncertainty is considered in the growth and harvesting terms. The concept of fuzzy mathematics is incorporated and the concerned terms are modified using trapezoidal fuzzy number (TrFN). The dynamical behaviour of the system is studied extensively. Some numerical simulations are carried out.

Keywords: Prey-predator; Trapezoidal fuzzy number; Harvesting; Global stability.

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1. INTRODUCTION

Mathematical modelling is one of the key scientific techniques to illustrate an ecological system. Researchers are involving to develop more and more accurate mathematical models to make a realistic representation of real world problems. Most of the models so developed are in the precise environment. These types of models are mainly governed by the differential equation. But in reality most of the biological or ecological parameters are not accurately measurable. For example, the growth of a particular species or the harvesting effort applied on any species can not determine uniquely.

So to handling these sensible biological parameters properly imprecise parameter should be more appropriate. Fuzzy or stochastic method can be used to overcome this impreciseness. In reality sometimes fuzzy method is well suited in compare to stochastic approaches. But in the fuzzy mathematics, we need to use a proper membership function to set up a mathematical model. In comparison with previous systems, fuzzy logic or the application of interval valued function is a more potential technique to apply any real life phenomenon. Fuzzy clustering method is one of the important procedures to handle the impreciseness in ecological parameter. In the mid sixty, Zadeh

(1965) was the first to introduce the concept of fuzzy set theory. It gains more and more attentions in the next decades. Almost, after a decade Dubois and Prade (1987) introduced the fuzzy numbers in 1978. They defined the fuzzy number as the fuzzy subset in the real axis. Then the introduction of fuzzy number and the development of mathematical model are growing rapidly in different field of applications. By the theory of fuzzy differential equations (Barros, Bassanezi and Tonelli (2000); Bede and Gal (2005); Ahamed, Hasan and Baets (2013); Ghosh, Kar and Legovic(2014)) we can resolve the systems with the uncertain initial states. It also helps us to understand the uncertain relationship between ecological parameters.

Fuzzy numbers are embodied with the several types of fuzzy numbers like trapezoidal or triangular fuzzy number. Sevastjanov *et al.* (2006) discussed some methods for the comparison of crisp and fuzzy interval optimization. Tapaswini *et al.* (2013) found the numerical solution of a fuzzy based predator prey system and discussed the efficiency of this method by depicting few interesting numerical results. Andrade *et al.* (2016) used the fuzzy modelling approach in planktonic network. Guo *et al.* (2012) introduced a class of fuzzy delay differential equation to check the existence of non oscillatory solutions. Recently, Nandi *et al.* (2018) analysed a Fuzzy epidemic model with saturated treatment and disease transmission rate and evaluated the fuzzy reproduction number.

In nature, the prey-predator interaction is one of the very common but important phenomena. Qualitative understanding of this introduction is very crucial in order to maintain the sustainability of the system and several research efforts are made to improving this understanding (Das and Guha (2016); Omar, Aous and Balasem (2015); Abbasbandy and Hajjari(2009); Ganguly and Kar (2019)). However very few analyses (Barros, Bassanezi and Tonelli(2000); Bezdec (1993)) are found based on fuzzy modelling approach. In addition, harvesting has also a strong impact on the dynamics of a prey predator system. The growing human needs more food and energy and it force to over harvest the biological resources. Thus, there is a global concern to protect the ecosystem. Several mathematical models concerning harvesting on prey-predator system can be found in (Kar and Matsuda (2007); Ghosh and Kar (2007); Ghosh, Kar and Legovic (2014)). Though mathematical analysis of exploited biological resources in fuzzy environment has received much attention in recent times, however existing literature is still somewhat inadequate (Chalco-Cano and Roman-Flores (2013); Das *et al.* (2014); Wang (2017)). Most of the biological parameters are not precise rather oscillate about some average so that the deterministic stationary state is not an absolutely fixed state, instead it is a fuzzy value. From this view point we modify a deterministic harvesting model to a fuzzy model.

Here, we adopted the trapezoidal fuzzy number for the specific two parameters of the model. Then it is mixed up with the interval arithmetic to study a new approach, "trapezoidal fuzzy interval arithmetic formulation". We thought this noble idea is more acceptable because, the parameter belongs to a specific interval with the least upper bound and the greatest lower bound. Furthermore, one may accept that in certain cases there is inherited uncertainty. So, it is needed to take the set of more

sensitive parameters that must lie within a particular region i.e. the few important parameter is fuzzy valued function.

Here we organized the paper in the following way. Fuzzy preliminary concepts are given in section 2. Crisp and fuzzy models are given in section 3. Some basic results including the positivity, boundedness and also the dynamical behaviour of the fuzzy systems are discussed in section 4. Global stability analyses of the fuzzy model are given in section 5. Some numerical works and concluding remarks are given in section 6.

2. FUZZY PRELIMINARY CONCEPTS

As there are no so much literature available on the application of fuzzy mathematics in population ecology, for the development of a prey predator model in fuzzy environment, some essential definitions and concepts are provided. Zadeh (1965) was the first mathematician who introduced the fuzzy sets and logic in a mathematical way to represent the ambiguous real life situations. Barua *et al.* (2014) mentioned that trapezoidal number is more acceptable in comparison with other.

A set is said to be fuzzy set if it comprises with a membership function. If we assume $\tilde{a}$ as a fuzzy set of a universal set Π and the corresponding membership function $\psi_{\tilde{a}}$, which is defined as $\psi_{\tilde{a}} : \Pi \rightarrow [0,1]$. It assigns each element x of Π , and the ordered pair $\{(x, \psi_{\tilde{a}}(x)) : x \in \Pi\}$ is denoted by $\tilde{a}$. Then, $\tilde{a}$ and $\psi_{\tilde{a}}(x)$ together defined as membership value or membership degree in the fuzzy set $\tilde{a}$.

A fuzzy set $\tilde{a} = (a_1, a_2, a_3)$ of the real line R together with a membership function $\psi_{\tilde{a}} : R \rightarrow [0,1]$ is called a triangular fuzzy number (TFN) if the corresponding membership function $\psi_{\tilde{a}}$ is defined as:

$$\psi_{\tilde{a}} = \begin{cases} \psi_{l\tilde{a}}(x) = \frac{x - a_1}{a_2 - a_1} \text{ for } a_1 \leq x \leq a_2, \\ \psi_{r\tilde{a}}(x) = \frac{a_3 - x}{a_3 - a_2} \text{ for } a_2 \leq x \leq a_3, \\ 0 \text{ otherwise.} \end{cases}$$

Similarly, a fuzzy set $\tilde{b} = (b_1, b_2, b_3, b_4)$ of the real line R together with its membership function $\psi_{\tilde{b}} : R \rightarrow [0,1]$ is called a trapezoidal fuzzy number, if membership function $\psi_{\tilde{b}}$ is defined by:

$$\psi_{\tilde{b}} = \begin{cases} 0, & x < b_1, \\ \frac{x - b_1}{b_2 - b_1}, & b_1 \leq x \leq b_2, \\ 1, & b_2 \leq x \leq b_3, \\ \frac{b_4 - x}{b_4 - b_3}, & b_3 \leq x \leq b_4, \\ 0, & x > b_4. \end{cases}$$

The α -cut of fuzzy number is defined by $[\tilde{a}]_{\alpha} = \{x \in R : \psi_{\tilde{a}}(x) \geq \alpha\}$. It is a crisp set. The nonempty closed and bounded interval number α -cut is denoted by

$$[\tilde{a}]_{\alpha} = [(\psi_{l\alpha})^{-1}\alpha, (\psi_{r\alpha})^{-1}\alpha].$$

In the closed interval of $[\tilde{a}]_{\alpha}$, $(\psi_{l\alpha})^{-1}\alpha$ and $(\psi_{r\alpha})^{-1}\alpha$ are the respective lower and upper bonds which are defined as,

$$\begin{aligned} (\psi_{l\alpha})^{-1}\alpha &= \inf\{x \in R : \psi_{\alpha} \geq \alpha\}, \\ (\psi_{r\alpha})^{-1}\alpha &= \sup\{x \in R : \psi_{\alpha} \geq \alpha\}. \end{aligned}$$

Every α -cut, $\alpha \in [0, 1]$ of A can be written by the closed interval

$$A_{\alpha} = [A_L(\alpha), A_R(\alpha)].$$

We have already discussed the interval arithmetic and it can be put down in any of the interval lying in the real axis by the function $i(p) = (a)^{1-p}(b)^p$, for $p \in [0, 1]$ for the interval $[a, b]$ and the above closed interval A_{α} can be written as

$$A_{\alpha} = (A_L(\alpha))^{1-p}(A_R(\alpha))^p.$$

Though we use this fuzzy mathematics to overcome the uncertainties in the real life situation it is still very tough to analyse a dynamical system or any other real life problem incorporating the fuzziness in the model. In a fuzzy approach, the imprecise parameters of interval arithmetic analysis are replaced by the fuzzy sets with a known membership function. To get a better approximation we use either a triangular or trapezoidal fuzzy number with a suitable membership function. To keep the model simple in fuzzy environment we usually consider one or two control parameters. To defuzzify a fuzzy number we use the graded mean value method for a triangular or the trapezoidal number.

Let us assume the pre-assigned $\lambda \in [0, 1]$ is the degree of optimism. Then we can express the graded mean value Chen (1985) of $\tilde{a}$ by the convex combination of the left and right integral values, such that,

$$I_{\lambda}(\tilde{a}) = \lambda I_r(\tilde{a}) + (1 - \lambda) I_l(\tilde{a})$$

where $I_l(\tilde{a})$ and $I_r(\tilde{a})$ are respectively denoted the left and right integral values of $\tilde{a}$. These are defined as,

$$I_l(\tilde{a}) = \int_0^1 (\psi_{l\alpha})^{-1} \alpha d\alpha \quad \text{and} \quad I_r(\tilde{a}) = \int_0^1 (\psi_{r\alpha})^{-1} \alpha d\alpha.$$

The triangular fuzzy number $\tilde{a} = (a_1, a_2, a_3)$ is defined in the total λ -integral value form as,

$$I_\lambda(\tilde{a}) = \left[\lambda \left(\frac{a_3 + a_2}{2} \right) + (1 - \lambda) \left(\frac{a_1 + a_2}{2} \right) \right], \quad 0 \leq \lambda \leq 1.$$

It can be directly fitted to any imprecise nature parameter set but if the trapezoidal number is used then another step is considered to compute the parameters. In our dynamical system, the uncertain parameters taken as trapezoidal fuzzy number. Now, $T = (T_1, T_2, T_3, T_4)$, the fuzzy trapezoidal number can be written as $T_\alpha = [T_L(\alpha), T_U(\alpha)]$, $\alpha \in [0, 1]$,

where $T_L(\alpha) = T_1 + \lambda(T_2 - T_1)$,

and $T_U(\alpha) = T_4 - \lambda(T_4 - T_3)$, $\lambda \in [0, 1]$

Then by the use of interval arithmetic we can write the trapezoidal fuzzy number

$$\begin{aligned} T_\alpha(T_1, T_2, T_3, T_4) &= (T_L(\alpha))^{1-p} (T_U(\alpha))^p, & p \in [0, 1] \\ &= (T_1 + \lambda(T_2 - T_1))^{1-p} (T_4 - \lambda(T_4 - T_3))^p, & p \in [0, 1], \lambda \in [0, 1] \end{aligned}$$

Now this interval arithmetic number, in lieu of the fuzzy trapezoidal number, is used as biological parameters in our dynamical system.

3. Crisp predator prey model

Let us consider $x(t)$ and $y(t)$ are represented the density of prey and predator population respectively at any time t . The intrinsic growth rate of prey population is denoted by r and carrying capacity is taken as K . Predator consumes prey at a rate α and σ ($0 < \sigma < 1$) is the conversion rate and δ denotes the predator's mortality rate. We also assume that there is an intraspecific competition among predator and this coefficient is γ . Thus the prey predator model takes the form:

$$\begin{aligned} \frac{dx}{dt} &= rx \left(1 - \frac{x}{K} \right) - \alpha xy \\ \frac{dy}{dt} &= \sigma \alpha xy - \delta y - \gamma y^2 \end{aligned} \tag{3.1}$$

It is also assumed that the predator population is subjected to harvesting based on catch per unit effort hypothesis. Thus the harvesting function is taken as $h = Ey$. Here the catchability coefficient is taken as 1, which does not change the qualitative behaviour of the system. Thus finally our model takes the form:

$$\begin{aligned}\frac{dx}{dt} &= rx \left(1 - \frac{x}{K}\right) - \alpha xy \\ \frac{dy}{dt} &= \sigma \alpha xy - \delta y - \gamma y^2 - Ey\end{aligned}\tag{3.2}$$

with $x(0) \geq 0, y(0) \geq 0$.

3.1 FUZZY PREDATOR PREY MODEL

The dynamical analysis of the model systems (3.2) is well known to the literature, but to the authors knowledge no attempt has made yet to study this model in fuzzy environment by considering some or all the parameters as imprecise parameters. Fuzzy numbers of the counterparts of the growth rate of prey species and the harvesting effort on predator species have a great impact to the dynamics of the system. Hence to incorporate environmental uncertainty, we consider the intrinsic growth rate r and the harvesting effort E as imprecise parameter. Though it would be more appropriate if all the parameters are taken as fuzzy number, but to keep model analyzable, we restrict the impreciseness for only two parameters. As the model is of logistic type, the prey biomass must have an upper bound. Similarly, harvesting effort must have an upper bound. Out of different fuzzy numbers, trapezoidal fuzzy number (Omar *et al.* (2015); Ghosh *et al.* (2014); Ahmad (2012)) is most suitable for these types of parameters. Thus, the system (3.2) takes the form,

$$\begin{aligned}\frac{dx}{dt} &= \hat{r}x \left(1 - \frac{x}{K}\right) - \alpha xy, \\ \frac{dy}{dt} &= \sigma \alpha xy - \delta y - \gamma y^2 - \hat{E}y.\end{aligned}\tag{3.3}$$

where $\hat{r} = [r_l, r_u] = (r_1 + \lambda(r_2 - r_1))^{1-p} (r_4 - \lambda(r_4 - r_3))^p$,

$\hat{E} = [E_l, E_u] = (E_1 + \lambda(E_2 - E_1))^{1-p} (E_4 - \lambda(E_4 - E_3))^p, (0 \leq \lambda \leq 1, 0 \leq p \leq 1)$

(r_1, r_2, r_3, r_4) and (E_1, E_2, E_3, E_4) are trapezoidal fuzzy numbers.

Trapezoidal number is constructed with an interval number with a suitable lower and upper bound, and the different values of λ between 0 to 1 can be reflected according to decision maker pessimistic, moderately pessimistic, and moderately optimistic or the optimistic standpoint. Considering a fuzzy parameter $\hat{r} = (r_1, r_2, r_3, r_4)$, if we choose $\lambda = 0$, as a pessimistic point of view, it takes r_1 for the lower bound of the trapezoidal number r_l and r_4 for the upper bound of trapezoidal number namely r_u . The higher degree of optimism indicates for a large value of λ . For $\lambda = 1$, it takes other two vertices of the trapezium r_2 and r_3 for the lower or upper bound of the trapezoidal

number $\hat{r}$. Then we deduce the interval $[r_l, r_u]$ with the interval function by varying a proper fraction p from 0 to 1, which reflects the each and every real number of the specified interval.

Thus finally the system becomes,

$$\begin{aligned}\frac{dx}{dt} &= (r_1 + \lambda(r_2 - r_1))^{1-p} (r_4 - \lambda(r_4 - r_3))^p x \left(1 - \frac{x}{K}\right) - \alpha xy, \\ \frac{dy}{dt} &= \sigma \alpha xy - \delta y - \gamma y^2 - (E_1 + \lambda(E_2 - E_1))^{1-p} (E_4 - \lambda(E_4 - E_3))^p y.\end{aligned}\quad (3.4)$$

Next, we investigate the dynamical behaviour of this model using trapezoidal fuzzy number.

4. DYNAMICAL ANALYSIS OF FUZZY MODEL

In this section our intention is to examine the system (3.4), the solution trajectories of which are confined in a positive region. Then we also study the boundedness of the fuzzy system. The following theorem 4.1 establishes the positivity of the system and theorem 4.2 establishes the boundedness of the system.

Theorem 4.1. *Every trajectories of the fuzzy system (3.4) are positive and belong to the positive quadrant.*

Proof. It directly follows from (Barros et al. (2000)).

Theorem 4.2. *All the solutions of the fuzzy system (3.4) are uniformly bounded in the region*

$$\Omega = \{(x, y) \in R_2^+ : \alpha x + y = K' + \varepsilon\} \text{ for any } \varepsilon > 0 \text{ as } t \rightarrow \infty.$$

Proof. For the sake of boundedness we construct a function Ω be such that $\Omega = \alpha x + y$, Then we have

$$\begin{aligned}\frac{d\Omega}{dt} &= \sigma \hat{r} x \left(1 - \frac{x}{K}\right) - \sigma \alpha xy + \sigma \alpha xy - \delta y - \gamma y^2 - \hat{E} y \\ &\leq \frac{\gamma \sigma K (\delta + \hat{r}) + \hat{E}^2 \hat{r}}{4\gamma \hat{r}} - \frac{\sigma \hat{r}}{K} \left(x - \frac{K(\delta + \hat{r})}{2r}\right)^2 - \gamma \left(y + \frac{\hat{E}}{2\gamma}\right)^2 - \Omega(t)\end{aligned}$$

$$\text{Hence } \frac{d\Omega}{dt} + d\Omega \leq K', \text{ where } K' = \frac{\gamma \sigma K (\delta + \hat{r}) + \hat{E}^2 \hat{r}}{4\gamma \hat{r}}$$

Integrating the above inequality, we get,

$$\Omega \leq \frac{d\Omega(0) - K'}{d} e^{-dt} + K'$$

So, from the above result, it is obvious that $\Omega \leq K'$, $\forall t$ and $\Omega \geq 0$. Thus we can conclude that $\Omega = \alpha x + y$ is bounded. So the desire system is bounded.

To determine the fixed point of the fuzzy system (3.4) we set $\dot{x}(t) = 0$, $\dot{y}(t) = 0$ get the following possible equilibrium points.

- i) The trivial equilibrium at $P_1(0, 0)$.
- ii) The predator free axial equilibrium at $P_2(K, 0)$.

and iii) The coexisting equilibrium point $P^*(x^*, y^*)$

where $x^* = \frac{[\hat{r} + \alpha(\delta + \hat{E})]K}{\hat{r} + K\sigma\alpha^2}$, $y^* = \frac{\alpha\hat{r}[K\alpha\sigma - \delta - \hat{E}]}{\hat{r} + K\sigma\alpha^2}$, here $\hat{r}$ and $\hat{E}$ are specified as

$$\hat{r} = (r_1 + \lambda(r_2 - r_1))^{1-p} (r_4 - \lambda(r_4 - r_3))^p, \quad \hat{E} = (E_1 + \lambda(E_2 - E_1))^{1-p} (E_4 - \lambda(E_4 - E_3))^p,$$

Here, $x^* > 0$, but $y^* > 0$ provided $\hat{E} < K\alpha\sigma - \delta$, that is when $(E_1 + \lambda(E_2 - E_1))^{1-p} (E_4 - \lambda(E_4 - E_3))^p < K\alpha\sigma - d$.

The Jacobian matrix of the dynamical system (3.4) at the interior equilibrium point is

$$J(x^*, y^*) = \begin{pmatrix} \frac{(r_1 + \lambda(r_2 - r_1))^{1-p} (r_4 - \lambda(r_4 - r_3))^p x^*}{K} & -\alpha x^* \\ \sigma\alpha y^* & -\gamma y^* \end{pmatrix}$$

Theorem 4.3.

- i) For the system (3.4) the trivial equilibrium point $P_1(0,0)$ is saddle.
- ii) The predator free equilibrium point $P_2(K,0)$ is always locally asymptotically stable if

$$\frac{\sigma\alpha K}{\delta + (E_1 + \lambda(E_2 - E_1))^{1-p} (E_4 - \lambda(E_4 - E_3))^p} < 1.$$

- iii) The dynamical system (3.4) is locally asymptotically stable around the interior equilibrium point $P^*(x^*, y^*)$ if

$$\delta < \min \left\{ \frac{K\gamma\sigma\alpha^2 (r_1 + \lambda(r_2 - r_1))^{1-p} (r_4 - \lambda(r_4 - r_3))^p - \gamma}{\alpha(1 + \gamma)} \right. \\ \left. - (E_1 + \lambda(E_2 - E_1))^{1-p} (E_4 - \lambda(E_4 - E_3))^p, K\sigma\alpha - (E_1 + \lambda(E_2 - E_1))^{1-p} (E_4 - \lambda(E_4 - E_3))^p \right\}$$

Proof.

- i) The variational matrix of the system (3.4) at P_1 is,

$$J_{P_1} = \begin{pmatrix} (r_1 + \lambda(r_2 - r_1))^{1-p} & 0 \\ (r_4 - \lambda(r_4 - r_3))^p & -\delta - (E_1 + \lambda(E_2 - E_1))^{1-p} (E_4 - \lambda(E_4 - E_3))^p \end{pmatrix}.$$

Clearly the eigen values are $(r_1 + \lambda(r_2 - r_1))^{1-p} (r_4 - \lambda(r_4 - r_3))^p (> 0)$ and $-(\delta + \hat{E}) < 0$. So the system has a saddle around $P_1(0,0)$.

- ii) The jacobian matrix of the system (3.4) evaluated at P_2 is

$$J_{P_2} = \begin{pmatrix} -(r_1 + \lambda(r_2 - r_1))^{1-p} & -\alpha K \\ (r_4 - \lambda(r_4 - r_3))^p & \sigma\alpha K - \delta - \hat{E} \end{pmatrix}.$$

Here the eigen values of the system are $-(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p < 0$ and $\sigma\alpha K - (\delta + \hat{E})$. The system is locally asymptotically stable $\sigma\alpha K < (\delta + \hat{E})$.

iii) The variational matrix of the system (3.4) evaluated around its interior equilibrium point

$$P^*(x^*, y^*) \text{ is, } J_{P^*} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\text{where, } a_{11} = \frac{(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p [\gamma(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p + \alpha(\delta + (E_1 + \lambda(E_2 - E_1))^{1-p}(E_4 - \lambda(E_4 - E_3))^p)]}{\gamma(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p + K\sigma\alpha^2},$$

$$a_{12} = -\frac{\alpha K [\gamma(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p + \alpha(\delta + (E_1 + \lambda(E_2 - E_1))^{1-p}(E_4 - \lambda(E_4 - E_3))^p)]}{\gamma(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p + K\sigma\alpha^2},$$

$$a_{21} = \frac{\sigma\alpha^2 (r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p [K\alpha\sigma - \delta - (E_1 + \lambda(E_2 - E_1))^{1-p}(E_4 - \lambda(E_4 - E_3))^p]}{\gamma(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p + K\sigma\alpha^2},$$

$$a_{22} = -\frac{\gamma\alpha (r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p [K\alpha\sigma - \delta - (E_1 + \lambda(E_2 - E_1))^{1-p}(E_4 - \lambda(E_4 - E_3))^p]}{\gamma(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p + K\sigma\alpha^2}.$$

The characteristic equation of the above variational matrix becomes,

$$\mu^2 - (a_{11} + a_{22})\mu + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

$$\gamma(r_1 + \lambda(r_2 - r_1))^{2-2p}(r_4 - \lambda(r_4 - r_3))^{2p} + \alpha(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p(1 + \gamma)(E_1 + \lambda(E_2 - E_1))^{1-p}(E_4 - \lambda(E_4 - E_3))^p - K\gamma\sigma\alpha^2\hat{r}$$

$$a_{11} + a_{22} = \frac{\gamma(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p + K\sigma\alpha^2}{\gamma(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p + K\sigma\alpha^2},$$

$$\alpha(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p$$

$$(\gamma(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p)$$

$$+ \alpha(\delta + (\delta + (E_1 + \lambda(E_2 - E_1))^{1-p}(E_4 - \lambda(E_4 - E_3))^p))$$

$$(K\sigma\alpha - (\delta + (E_1 + \lambda(E_2 - E_1))^{1-p}(E_4 - \lambda(E_4 - E_3))^p))$$

$$a_{11}a_{22} - a_{12}a_{21} = \frac{(\gamma(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p + K\sigma\alpha^2)}{(\gamma(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p + K\sigma\alpha^2)^2},$$

Let the roots of the above equation are μ_1 and μ_2 . Then from the relation between roots and coefficient, we get

$$\mu_1 + \mu_2 = \frac{\gamma(r_1 + \lambda(r_2 - r_1))^{2-2p}(r_4 - \lambda(r_4 - r_3))^{2p} + \alpha(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p(1 + \gamma)(d + (E_1 + \lambda(E_2 - E_1))^{1-p}(E_4 - \lambda(E_4 - E_3))^p) - K\gamma\sigma\alpha^2\hat{r}}{\gamma(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p + K\sigma\alpha^2}, \quad (4.1)$$

$$\mu_1\mu_2 = \frac{\alpha(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p(\gamma(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p) + \alpha(d + (\delta + (E_1 + \lambda(E_2 - E_1))^{1-p}(E_4 - \lambda(E_4 - E_3))^p)) (K\sigma\alpha - (\delta + (E_1 + \lambda(E_2 - E_1))^{1-p}(E_4 - \lambda(E_4 - E_3))^p)) (\gamma(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p) + K\sigma\alpha^2}{(\gamma(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p + K\sigma\alpha^2)^2}. \quad (4.2)$$

For stability of the system the eigen values μ_1 and μ_2 are must be negative, then from the above relation (4.1) and (4.2) we must have $\mu_1 + \mu_2 < 0$ and $\mu_1\mu_2 > 0$. Then we have $\gamma(r_1 + \lambda(r_2 - r_1))^{2-2p}(r_4 - \lambda(r_4 - r_3))^{2p} + \alpha(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p(1 + \gamma)(\delta + (E_1 + \lambda(E_2 - E_1))^{1-p}(E_4 - \lambda(E_4 - E_3))^p) < K\gamma\sigma\alpha^2(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p$ because always $\gamma(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p + K\sigma\alpha^2 > 0$. From the second relation we get,

$$\delta < \frac{K\gamma\sigma\alpha^2(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p - \gamma}{\alpha(1 + \gamma)} - (E_1 + \lambda(E_2 - E_1))^{1-p}(E_4 - \lambda(E_4 - E_3))^p \quad (4.3)$$

Again from the relation (4.2) we know that $\mu_1\mu_2 > 0$, which implies that

$$\begin{aligned} & \alpha(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p(\gamma(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p) \\ & + \alpha(d + (\delta + (E_1 + \lambda(E_2 - E_1))^{1-p}(E_4 - \lambda(E_4 - E_3))^p)) \\ & (K\sigma\alpha - (\delta + (E_1 + \lambda(E_2 - E_1))^{1-p}(E_4 - \lambda(E_4 - E_3))^p)) \\ & (\gamma(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p) + K\sigma\alpha^2 > 0 \\ & \text{or, } (K\sigma\alpha - (\delta + (E_1 + \lambda(E_2 - E_1))^{1-p}(E_4 - \lambda(E_4 - E_3))^p)) > 0 \\ & \text{or, } \delta < K\sigma\alpha - (E_1 + \lambda(E_2 - E_1))^{1-p}(E_4 - \lambda(E_4 - E_3))^p \end{aligned} \quad (4.4)$$

From the above two inequality relation (4.3) and (4.4) we have

$$\delta < \min \left\{ \frac{K\gamma\sigma\alpha^2(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p - \gamma}{\alpha(1 + \gamma)} \right. \\ \left. - (E_1 + \lambda(E_2 - E_1))^{1-p}(E_4 - \lambda(E_4 - E_3))^p, K\sigma\alpha - (E_1 + \lambda(E_2 - E_1))^{1-p}(E_4 - \lambda(E_4 - E_3))^p \right\}$$

Hence the proof.

5. GLOBAL STABILITY ANALYSIS

Our interest in this section is to find out the sufficient condition of global stability of the fuzzy system (3.4) by setting a suitable Lyapunov function as follows:

$$V(x, y) = \sigma \left[(x - x^*) - x^* \log \left(\frac{x}{x^*} \right) \right] + \left[(y - y^*) - y^* \log \left(\frac{y}{y^*} \right) \right], \quad (5.1)$$

The time derivative of the above equation along the trajectories of (3.4) is

$$\frac{dV}{dt} = \sigma(x - x^*) \left[\hat{r} \left(1 - \frac{x}{K} \right) - \alpha y \right] + (y - y^*) [\sigma \alpha x - \delta - \gamma y - \hat{E}] \\ \text{or, } \frac{dV}{dt} = - \frac{\sigma(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p}{K} (x - x^*)^2 - \sigma \alpha (x - x^*) (y - y^*) \\ + \sigma \alpha (y - y^*) (x - x^*) - \gamma (y - y^*)^2 \\ = -X^T A X < 0$$

where $X^T = [(x - x^*) \quad (y - y^*)]$ and $A = \begin{pmatrix} \frac{\sigma(r_1 + \lambda(r_2 - r_1))^{1-p}(r_4 - \lambda(r_4 - r_3))^p}{K} & 0 \\ 0 & \gamma \end{pmatrix}$. Here the

matrix A is always positive definite without embedding any condition. Thus $dV/dt < 0$ and the global stability of the system (3.2) around the interior equilibrium point $E^*(x^*, y^*)$ comes automatically.

6. NUMERICAL SIMULATION AND CONCLUDING REMARKS

In this section we perform some simulation works to our fuzzy system (3.4). The two parameters, the intrinsic growth rate $\hat{r} = (r_1, r_2, r_3, r_4)$ and harvesting effort to harvesting the predator population $\hat{E} = (E_1, E_2, E_3, E_4)$ set as trapezoidal fuzzy number. For the simulation purpose, we consider the data set as

$$r_1 = 0.7, r_2 = 0.8, r_3 = 1.05, r_4 = 1.15, K = 70, \alpha = 0.85, \sigma = 0.265, d = 0.001, \gamma = 0.05, E_1 = 0.35, \\ E_2 = 0.45, E_3 = 0.78, E_4 = 0.95.$$

Now varying the proper fraction p and the corresponding value of λ from 0 to 1, we have depicted all 4×4 phase plots of prey and predators. Here we incorporated the four figures for $p = 0.0, \lambda = 0.0, 0.4, 0.7, 1.0$; $p = 0.4, \lambda = 0.0, 0.4, 0.7, 1.0$; $p = 0.7, \lambda = 0.0, 0.4, 0.7, 1.0$ and $p = 1.0, \lambda = 0.0, 0.4, 0.7, 1.0$.

In the Fig.1, four phase plots are drawn, for the fixed p but λ varies from 0 to 1, we get different the stable diagram. The only difference in the above diagrams are, for a fixed value of p , the system stable becomes at relatively high biomass system for the enhancement of λ . This feature continues

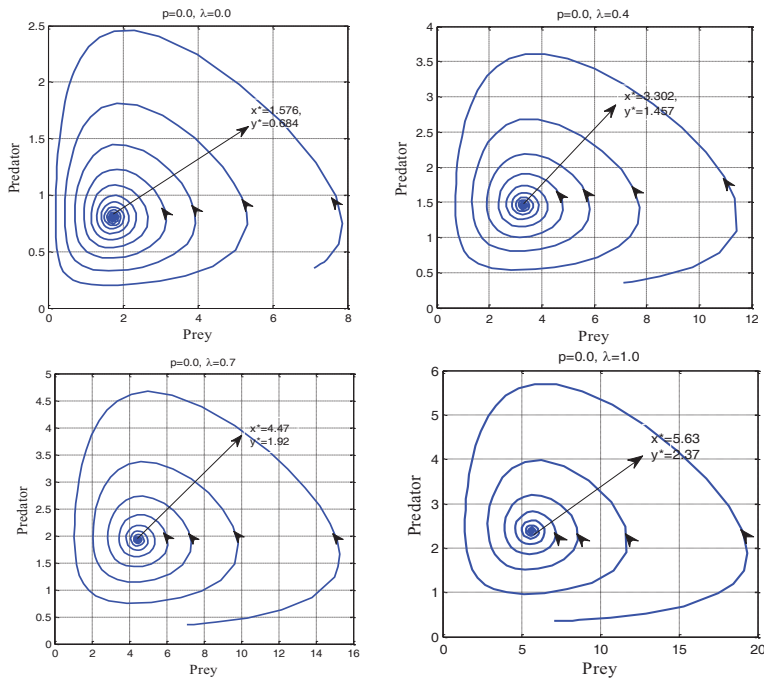
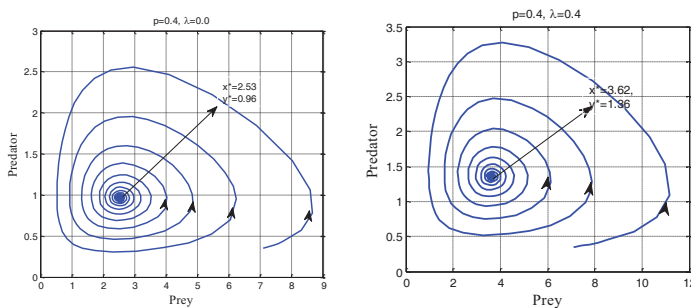


Figure.1. Various phase plot of prey population and predator population with $p = 0.0$ and $\lambda = 0.0, \lambda = 0.4, \lambda = 0.7, \lambda = 1.0$ respectively.



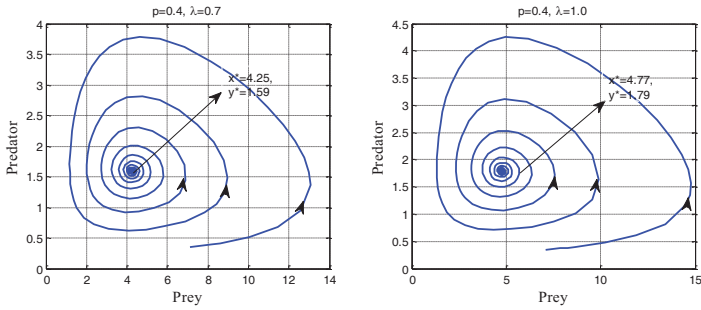


Figure.2. Various phase plot of prey and predator population with $p = 0.4$ and $\lambda = 0.0, \lambda = 0.4, \lambda = 0.7, \lambda = 1.0$.

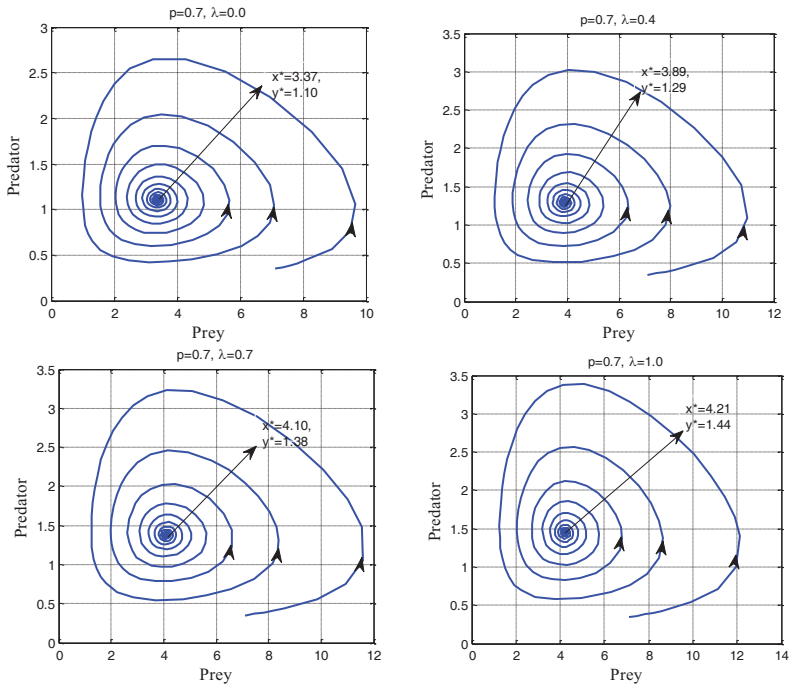
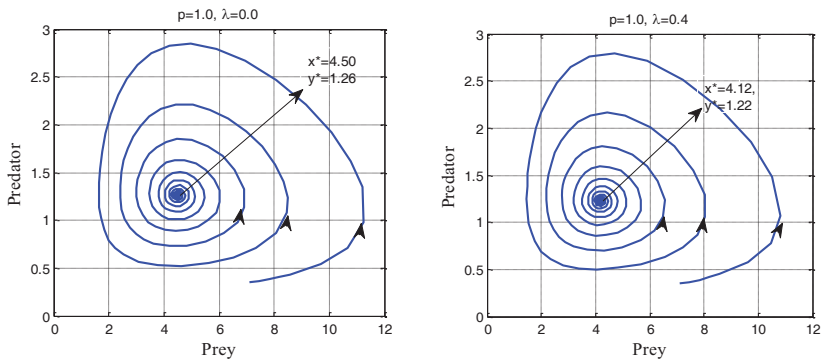


Figure.3. Various phase plot of prey and predator population with $p = 0.7$ and $\lambda = 0.0, \lambda = 0.4, \lambda = 0.7, \lambda = 1.0$.



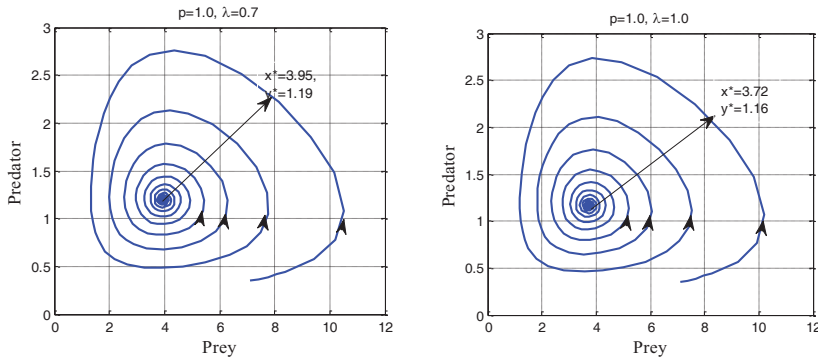


Figure.4. Various phase plot of prey and predator population with $p = 1.0$ and $\lambda = 0.0, \lambda = 0.4, \lambda = 0.7, \lambda = 1.0$.

for $p = 0.0$ to $p = 0.7$. Corresponding trajectories are clearly depicted in the Fig.2 and Fig.3 respectively. However, the reverse nature is shown in the Fig.4 for $p = 1.0$. Here the stability occurs at relatively low population level for the high value of λ . Moreover, we see that the stable fixed point of each trajectory is same as our analytical findings which are calculated in the mathematica.

The details of the analytical results are given in table 1.

Different values of p	Different values of λ	Interior Equilibrium point x^* and y^*	Eigen values	Nature of stability
$p = 0.0$	$\lambda = 0.0$ $\lambda = 0.4$ $\lambda = 0.7$ $\lambda = 1.0$	$x^* = 1.7365 \quad y^* = 0.8031$ $x^* = 3.3029 \quad y^* = 1.457$ $x^* = 4.4721 \quad y^* = 1.9272$ $x^* = 5.638 \quad y^* = 2.379$	$\mu = -0.024 \pm 0.47i$ $\mu = -0.027 \pm 0.51i$ $\mu = -0.029 \pm 0.54i$ $\mu = -0.031 \pm 0.57i$	Stable focus
$p = 0.4$	$\lambda = 0.0$ $\lambda = 0.4$ $\lambda = 0.7$ $\lambda = 1.0$	$x^* = 2.536 \quad y^* = 0.968$ $x^* = 3.6268 \quad y^* = 1.3613$ $x^* = 4.2541 \quad y^* = 1.5942$ $x^* = 4.7737 \quad y^* = 1.794$	$\mu = -0.034 \pm 0.62i$ $\mu = -0.036 \pm 0.64i$ $\mu = -0.038 \pm 0.66i$ $\mu = -0.04 \pm 0.68i$	Stable focus
$p = 0.7$	$\lambda = 0.0$ $\lambda = 0.4$ $\lambda = 0.7$ $\lambda = 1.0$	$x^* = 3.3765 \quad y^* = 1.1095$ $x^* = 3.8971 \quad y^* = 1.2930$ $x^* = 4.1024 \quad y^* = 1.3824$ $x^* = 4.2167 \quad y^* = 1.4493$	$\mu = -0.0456 \pm 0.78i$ $\mu = -0.0455 \pm 0.76i$ $\mu = -0.0454 \pm 0.74i$ $\mu = -0.0453 \pm 0.72i$	Stable focus
$p = 1.0$	$\lambda = 0.0$ $\lambda = 0.4$ $\lambda = 0.7$ $\lambda = 1.0$	$x^* = 4.5029 \quad y^* = 1.2659$ $x^* = 4.1296 \quad y^* = 1.2276$ $x^* = 3.9597 \quad y^* = 1.1987$ $x^* = 3.7268 \quad y^* = 1.1695$	$\mu = -0.062 \pm 0.97i$ $\mu = -0.057 \pm 0.91i$ $\mu = -0.054 \pm 0.87i$ $\mu = -0.051 \pm 0.84i$	Stable focus

Table-1

The vertices of the trapezoidal fuzzy number of the prey and the predator population can be written as $[1.7365, 2.536, 4.7737, 5.638]$ and $[0.8031, 0.968, 1.927, 2.379]$ respectively.

Here, we discuss the dynamical behaviour of two dimensional prey-predator model with predator harvesting. It is assumed that intrinsic growth rate and harvesting effort are fuzzy number. To get more accurate, precise and perfect result, we used the fuzzy interval arithmetic system for future cases. Following, Barua *et al.* (2014), we have introduced that the trapezoidal fuzzy number for the sake of simplicity. Then some basic properties like positivity and boundedness of the system are determined.

Here we propose a new thought of fuzzy number combining with interval arithmetic. Some authors have used either the interval valued function or the fuzzy number, but in this approach we use an uncertain fixed value is used. Though our paper is not a case study, but this result can be applied to fisheries and forestry. In a more general way, we can extend the total system in fuzzy environment. This approach shall be more convenient and acceptable for its reality in the nature. Both local and global stability of the system is studied. Finally, we may conclude that fuzzy based model system would be a new research destination on mathematical ecology.

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