

Dynamical study of effects of herbivores on corals and algae in a coral reef ecosystem

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ABSTRACT

It is well known that coral reefs are at a high risk of destruction from terrestrial runoff, various anthropogenic activities and pollution in marine ecosystem. The growth of coral reef and macro algae depend on the activities of herbivores on it. Here, we introduce a mathematical model to explain the effect of activities of herbivores on growth of coral reef and macro algae in a coral reef ecosystem and investigate the time dependent sensitivity index of parameters on different species. A detailed analysis of the model reveals that the system undergoes a Hopf-bifurcation when the rate of decline of herbivorous fishes pass through its critical value. Competition for space and light between coral and algae in herbivore free system is derived conducting analytical study of the model and coexistence between corals and algae is also illustrated using numerical simulations.

Keywords: Hopf-bifurcation, Competitiveness, Sensitivity analysis, Sensitivity index, Co-existence.

2000 Mathematics Subject Classification: 92B05, 37N25.

1 Introduction

Coral reefs are one of the most diverse ecosystems in the world and also provide a well-studied example of a Complex ecosystems with nonlinear interactions. Corals occupy less than 0.2% of the sea floor. Yet they cover more than 150,000 kilometers of coastline spread across more than 100 countries and territories. More than 275 million people live within 10 kilometers of the coast and within 30 kilometers of the coral ecosystem. One-eighth of the worlds population, about 850 million people live within 100 kilometers of the coral ecosystem and can take advantage of the ecological services provided by the coral reefs (Cinner,

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McClanahan, Daw, Graham, Maina, Wilson and Hughes, 2009; Burke, Reyta, Spalding and Perry, 2011; Costanza, De Groot, Sutton, Van der Ploeg, Anderson, Kubiszewski, Farber and Turner, 2014). However, today the coral reefs are in great danger, as they are on the verge of extinction for a variety of reasons that have resulted in the loss of about 50% of coral reefs since the early 1980s (Bruno and Selig, 2007; Côté, Gill, Gardner and Watkinson, 2005). One of the important reasons causing loss of corals is the anthropogenic stresses. Conditions arising out of anthropogenic activities such as climate change, pollution, sedimentation, and the overexploitation of fish stocks are the primary causes of reef degradation over the last several decades (Hoegh-Guldberg, Mumby, Hooten, Steneck, Greenfield, Gomez, Harvell, Sale, Edwards, Caldeira et al., 2007; Keller, Gleason, McLeod, Woodley, Airamé, Causey, Friedlander, Grober-Dunsmore, Johnson, Miller et al., 2009). Overfishing of herbivorous fishes or eutrophication of seawater, which changes the local environment of the coral reefs into an environment that are more suitable for macroalgae to grow than that for corals, may cause the phase shift from coral- to macroalgae-dominated communities (Pal and Bhattacharyya, 2017; Hughes, 1994; Bellwood, Hughes, Folke and Nyström, 2004). Existence of hysteresis through saddle node bifurcation and transcritical bifurcation is also observed in coral reef ecosystems in presence of micro algal toxicity and overfishing (Bhattacharyya and Pal, 2015; Pal and Bhattacharyya, n.d.). Since microalgae are superior space competitors than corals macroalgae overgrow corals where the conditions are favorable for the growth of macroalgae in a reef (Box and Mumby, 2007). The largest group of coral reef fishes that feed on microscopic and macroscopic algae include parrotfishes, damselfishes, rabbitfishes, and surgeonfishes. So, the presence of these herbivorous fishes in the reef ecosystem in turn helps in creating favorable condition for the growth of corals. In addition, harvesting of invasive *Pterios volitans* is also considered to be an important factor to maintain the stability of the coral reef ecosystem (Maji, Bhattacharyya and Pal, 2016).

Competition for space and light are two important factors that determine the structure and composition of Benthic communities on coral reefs (Lang and Chornesky, 1990; Karlson, 2002). Competitive superiority of algae due to reduced herbivory and increased nutrient availability results into degradation of coralreef ecosystem. Specially, competition between hard corals and benthic algae becomes a very important factor during Phase shift, where macroalgae become dominant over reef building corals in coral reefs that were previously dominated by reef building corals (McCook, 1999; Ansell, Gibson, Barnes and Press, 1998; LaPointe, 1989).

We have constructed a model which have been highlighted through out the manuscript. In this model, we have taken three species i.e algae, herbivores and corals, where growth of algae and corals follow logistic growth rate (Solomonovich, Bica, Gao and Hailu, 2018) and herbivores feed on algae (Tolentino-Pablico, Bailly, Froese and Elloran, 2007; Kopp, Bouchon-Navaro, Cordonnier, Haouisée, Louis and Bouchon, 2010) following Holling type-II functional response.

The rest of this manuscript is as follows. In the next section, we have formulated the model. The mathematical properties of the model is demonstrated in section-3. In section-4, the system undergoes a Hopf-bifurcation when the number of herbivores decline. In section-5,

herbivore free system becomes a competition model between corals and macro algae. In section-6, numerical analysis has been conducted to confirm the analytical calculation and to demonstrate the coexistence stability and finally, a conclusion has been drawn from the whole study.

2 The mathematical model

Let $x(t)$, be the proportion of algal cover at time t , $z(t)$ be the proportion of coral cover, $y(t)$ be the herbivorous fish density at any time $t > 0$.

The following assumptions are made in deriving the mathematical model:

(H1) : In the absence of corals and herbivorous fishes the growth of macro algae population follows logistic law and also the growth of coral cover follows logistic law in the absence of algae. r_1 and r_2 are the intrinsic growth rate of algae and coral respectively.

(H2) : k_1 and k_2 are the carrying capacity of algae and coral respectively.

(H3) : Interaction between herbivores and algae follows Holling-II functional response with maximal growth rate a_2 and half saturation constant a_3 . A brief discussion regarding this was also published (Murray, Stillman, Gozlan and Britton, 2013).

(H4) : Algae and corals are competing for light and space which is shown here by the coefficient a_1 and c_1 .

(H5) : d_1 is the natural death rate of herbivorous fishes.

(H6) : b_2 is a density-dependent coefficient and b_1 is the growth efficiency of herbivorous fishes.

$$\begin{aligned}\frac{dx}{dt} &= r_1x \left(1 - \frac{x}{K_1}\right) - a_1xz - \frac{a_2yx}{(a_3 + x)}, \\ \frac{dy}{dt} &= \frac{b_1a_2xy}{(a_3 + x)} - b_2y^2 - d_1y, \\ \frac{dz}{dt} &= r_2z \left(1 - \frac{z}{K_2}\right) - c_1xz.\end{aligned}\tag{2.1}$$

3 Adimensionalization of the problem

Let us change the variables of the system (2.1) to non dimensional ones (Morozov and Arashkevich, 2008) by substituting $\bar{x} = \frac{x}{k_1}$, $\bar{y} = \frac{a_2y}{r_1k_1}$, $\bar{z} = \frac{z}{k_2}$, $\bar{t} = \frac{t}{r_1}$, $\bar{a}_1 = \frac{a_1k_2}{r_1}$, $\bar{a}_3 = \frac{a_3}{k_1}$, $\bar{a}_2 = \frac{b_1a_2}{r_1}$, $\bar{b}_2 = \frac{b_2k_1}{a_2}$, $\bar{d}_1 = \frac{d_1}{r_1}$, $\bar{r}_2 = \frac{r_2}{r_1}$, and $\bar{c}_1 = \frac{c_1k_1}{r_1}$.

Under these substitutions the system (2.1) reduces to (for simplicity we have used the parameters without bar):

$$\begin{aligned}\frac{dx}{dt} &= x(1-x) - a_1xz - \frac{xy}{(a_3+x)}, \\ \frac{dy}{dt} &= \frac{a_2xy}{(a_3+x)} - b_2y^2 - d_1y, \\ \frac{dz}{dt} &= r_2z(1-z) - c_1xz.\end{aligned}\tag{3.1}$$

$$\frac{dX}{dt} = f(X)\tag{3.2}$$

where $X = \begin{bmatrix} x & y & z \end{bmatrix}^T$ and $f(X) = \begin{bmatrix} xF & yG & zH \end{bmatrix}^T$ with
 $F = (1-x) - a_1z - \frac{y}{(a_3+x)}$
 $G = \frac{a_2x}{(a_3+x)} - b_2y - d_1$
 $H = r_2(1-z) - c_1x$
 $x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$.

3.1 Boundedness

To prove boundedness, we define a variable $U = x + y + z$,
then from system (3.1) we have :

Since $x(t)$ and $z(t) \leq 1$ as $t \rightarrow \infty$, it follows that corresponding to $\epsilon > 0$ there exists $t_\epsilon > 0$ such that $x(t) \leq 1 + \epsilon_1$, $z(t) \leq r_2 + \epsilon_2$, for all $t > t_\epsilon$. Where $\epsilon = \max\{\epsilon_1, \epsilon_2\}$

$$\frac{dU}{dt} + DU < (1 + r_2).$$

Where $D = \min\{1, d_1, r_2\}$

Let $U(t)$ be a solution of $\frac{dU}{dt} + DU < (1 + r_2)$ satisfying $U(0) = x(0) + y(0) + z(0)$.

Then $U(t) = \frac{1+r_2}{D} + \{x(0) + y(0) + z(0)\} \exp(-Dt) \rightarrow \frac{1+r_2}{D}$ as $t \rightarrow \infty$.

By comparison, it follows that $x(t) + y(t) + z(t) \leq \frac{1+r_2}{D}$ as $t \rightarrow \infty$.

Therefore we can say that the system (3.1) is bounded.

3.2 Equilibria and their Stability

In this section, we find all possible equilibria and their stabilities. By considering the system (3.1), the following equilibria are given below and as follow:

1. The trivial state: $O=(0,0,0)$.

2. The axial states: $R=(0,0,1)$, $Q=(1,0,0)$. However, Coral, algae free system does not exist.

3. The boundary states: $E=(x_E, 0, z_E) = (\frac{a_1-r_2}{a_1c_1-r_2}, 0, \frac{c_1-1}{a_1c_1-r_2})$ exists if any one of the conditions is satisfied

$$\{a_1 < \frac{r_2}{c_1}, c_1 < r_2 \text{ and } a_1 < 1\} \text{ or } \{a_1 > \frac{r_2}{c_1}, c_1 > r_2 \text{ and } a_1 > 1\}.$$

$$F=(x_F, y_F, 0).$$

Where $y_F = (a_3 + x_F)(1 - x_F)$ and x_F is the root of the equation

$$b_2(1 - x_F)(a_3 + x_F)^2 + d_1(a_3 + x_F) - a_2x_F = 0$$

This equation gives a positive root i.e the system exists if the condition is satisfied

$$\frac{1}{2} < a_3 < 1 + \sqrt{\left(\frac{d_1-a_2}{b_2}\right)} \text{ or } a_3 < \frac{1}{2}.$$

4. The interior steady state: $E^* = (x^*, y^*, z^*)$ is the solution of $xF(x, y, z) = yG(x, y, z) = zH(x, y, z) = 0$,

so that y^* and z^* satisfy

$$y^* = \frac{a_2x^*}{b_2(a_3+x^*)} - \frac{d_1}{b_2}, z^* = (1 - \frac{c_1x^*}{r_2})$$

and x^* is a solution of the cubic equation:

$$Ax^3 + Bx^2 + Cx + D = 0. \text{ Where}$$

$$A = (\frac{a_1c_1}{r_2} - 1), B = 2a_3(\frac{a_1c_1}{r_2}) - 1 + (1 - a_1)$$

$$C = a_3^2(\frac{a_1c_1}{r_2} - 1) + 2a_3(1 - a_1) - \frac{a_2}{b_2} + \frac{d_1}{b_2}, D = a_3^2(1 - a_1) + \frac{d_1a_3}{b_2}.$$

$$x^* \text{ is unique positive root if } a_1 < \frac{r_2}{c_1}.$$

3.3 Equilibrium analysis

Now, the local stability of the trivial state O , the axial states R , Q and the boundary states E and F are analyzed by showing the eigenvalues of the Jacobian matrix of the system (3.1)

$$J = \begin{pmatrix} F + xF_x & xF_y & xF_z \\ yG_x & G + yG_y & yG_z \\ zH_x & zH_y & H + zH_z \end{pmatrix},$$

about different equilibria. By espying at eigenvalues of J , we can easily check that O is an unstable node, whereas the axial solutions $Q(1,0,0)$ represents a macroalgae-dominated states and is locally asymptotically stable if $d_1 > \frac{a_2}{a_3+1} = A_0$ and $r_2 < c_1$, and $R(0,0,1)$ gives coral-dominated states and is locally asymptotically stable if $d_1 > A_0$ and $a_1 > 1$.

Now we consider the boundary points:

For boundary point E we have the eigenvalues are

$$\lambda_1 = \frac{a_2(a_1-r_2)}{a_3(a_1c_1-r_2)+(a_1-r_2)} - d_1 \text{ and}$$

other two eigenvalues are the root of

$$\lambda^2 - (a_{11} + a_{33})\lambda + (a_{11}a_{33} - a_{13}a_{31}) = 0. \text{ Where}$$

$$a_{11} = -x_E, a_{13} = -a_1x_E, a_{31} = -c_1z_E, a_{33} = -r_2z_E$$

Therefore the system is asymptotically stable if $a_1 < \frac{r_2}{c_1}$ and is unstable if $a_1 > \frac{r_2}{c_1}$

For the boundary point F, the eigenvalues are

$\lambda_1 = r_2 - c_1 x_F$ and λ_2, λ_3 are the root of the equation:

$-\lambda^2 + (a_{11} + a_{22})\lambda + a_{12}a_{21} - a_{11}a_{22} = 0$ Where

$$a_{11} = -x_F + \frac{x_F y_F}{(a_3 + x_F)^2}, a_{22} = -b_2 y_F, a_{12} = \frac{x_F}{a_3 + x_F}, a_{21} = \frac{a_2 a_3 y_F}{(a_3 + x_F)^2}$$

The system is asymptotically stable if

$$x_F \left\{ \frac{1}{(a_3 + x_F)^2} - \frac{1}{y_F} \right\} < b_2 < \frac{a_2 a_3}{(a_3 + x_F) y_F + (a_3 + x_F)^2}.$$

Next, we analyze the stability of interior equilibrium point $E^* = (x^*, y^*, z^*)$. The Jacobian matrix at E^* is the form

$$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix},$$

Where,

$$a_{11} = -x^* + \frac{x^* y^*}{(a_3 + x^*)^2}, a_{12} = -\frac{x^*}{(a_3 + x^*)}, a_{13} = -a_1 x^*, a_{21} = \frac{a_2 a_3 y^*}{(a_3 + x^*)^2}, a_{22} = -b_2 y^*, a_{31} = -c_1 z^*, a_{33} = -r_2 z^*.$$

The eigenvalues are satisfied by the equation

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \quad (3.3)$$

Where

$$A = -(a_{11} + a_{22} + a_{33})$$

$$B = a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11} - a_{12}a_{21} - a_{13}a_{31}$$

$$C = a_{12}a_{21}a_{33} + a_{13}a_{31}a_{22} - a_{11}a_{22}a_{33}$$

By the Routh-Hurwitz's criterion for stability it follows that the system is locally asymptotically stable at E^* if $A > 0, C > 0$ and $AB - C > 0$.

Therefore the system is asymptotically stable if the conditions are satisfied

$$y^* < (a_3 + x^*)^2, \frac{\sqrt{(a_1 c_1 x^* z^*)}}{y^*} < b_2 < \frac{a_2 a_3}{y^*}. \quad (3.4)$$

4 Hopf Bifurcation

The system (2.1) undergoes a Hopf bifurcation at the equilibrium point E^* when the parameter b_2 passes b_2^* . The same thing applies to the parameter d_1 . In this section, we will prove this in

the case of b_2 .

Let $\lambda(b_2) = \xi(b_2) + i\eta(b_2)$ be the root of the equation (3.3). Then putting this value we obtain the real and imaginary part respectively as

$$\begin{aligned}\xi^3 - 3\xi\eta^2 + A\xi^2 - A\eta^2 + B\xi + C &= 0 \\ 3\xi^2\eta - \eta^3 + 2A\xi\eta + B\eta &= 0.\end{aligned}\quad (4.1)$$

. A necessary condition for a change in stability of E^* is the the characteristic equation (3.3) having purely imaginary roots. So we have $\xi(b_2^*) = 0$ and equation (4.1) gives

$$\begin{aligned}-A\eta^2 + C &= 0 \\ -\eta^3 + B\eta &= 0.\end{aligned}\quad (4.2)$$

. Solving these we get $\eta = \eta^* = \sqrt[3]{B}$ and $AB - C = 0$.

Therefore the equation 3.3 can be written as

$$(\lambda^2 + B)(\lambda + A) = 0$$

It is clear that two roots of the equation (3.3) at $b_2 = b_2^*$ are

$$\lambda_{1,2} = \pm i\sqrt[3]{B} \text{ and } \lambda_3 = -A$$

Now $\eta^* > 0$ and the root λ_3 is negative using the condition 3.4.

We can confirm the transversality condition

$$\frac{d[Re\lambda(b_2)]}{db_2} \Big|_{b_2=b_2^*} \neq 0$$

Differentiating equation 4.1 with respect to b_2 and substituting $\xi = 0$, $\eta = \eta^*$ at $b_2 = b_2^*$, we have

$$\begin{aligned}U \frac{d[Re(\lambda(b_2))]}{db_2} + V \frac{d[Im(\lambda(b_2))]}{db_2} &= M \\ -V \frac{d[Re(\lambda(b_2))]}{db_2} + U \frac{d[Im(\lambda(b_2))]}{db_2} &= N\end{aligned}\quad (4.3)$$

Where,

$$U = (-3\eta^2 + B)|_{\eta=\eta^*, b_2=b_2^*}$$

$$V = -2A\eta|_{\eta=\eta^*, b_2=b_2^*}$$

$$M = A'\eta^2 - C'|_{\eta=\eta^*, b_2=b_2^*}$$

$$N = \eta^3 - B'\eta|_{\eta=\eta^*, b_2=b_2^*}$$

Solving the equation 4.3, we get

$$\frac{d[Re\lambda(b_2)]}{db_2} \Big|_{b_2=b_2^*} = \frac{MU - NV}{U^2 + V^2}$$

Now we say that $\frac{d[Re\lambda(b_2)]}{db_2} \Big|_{b_2=b_2^*} \neq 0$ science

$$\{(A'\eta^2 - C')(B - 3\eta^2) + 2A\eta(\eta^3 - B'\eta)\}|_{\eta=\eta^*, b_2=b_2^*} \neq 0$$

5 Herbivore free system

When herbivore fishes do not exist in the system (2.1) the system becomes two dimensional, i.e we get:

$$\begin{aligned}\frac{dx}{dt} &= r_1x \left(1 - \frac{x}{K_1}\right) - a_1xz \equiv F_1(x, z) \\ \frac{dz}{dt} &= r_2z \left(1 - \frac{z}{K_2}\right) - c_1xz \equiv F_2(x, z).\end{aligned}\quad (5.1)$$

$F_1(x, z)$ and $F_2(x, z)$ can be written as:

$$\begin{aligned}F_1(x, z) &= r_1x - g_{11}x^2 - g_{12}zx, \\ F_2(x, z) &= r_2z - g_{21}z^2 - g_{22}xz\end{aligned}\quad (5.2)$$

Where

$$g_{11} = \frac{r_1}{k_1}, g_{12} = a_1, g_{21} = c_1 \text{ and } g_{22} = \frac{r_2}{k_2}.$$

The system (5.1) is competitive science $\frac{\delta F_1}{\delta z} = -a_1x$ and $\frac{\delta F_2}{\delta x} = -c_1z$ is negative (Hirsch, 1982).

6 Numerical simulations

In this section numerical simulations have been conducted to observe the dynamics of the system (2.1) by using the set of parameters given in Table-1 (Blackwood, Hastings and Mumby, 2012; Fattahpour, Zangeneh and Wang, 2019), although the values of some parameters have been changed from time to time, which is clearly mentioned.

Table 1: Parameter values in the system (2.1)

Variables/	Descriptions	Values	Dimension
r_1	Intrinsic growth rate of algae	0.7	1/time
k_1	carrying capacity of algae	0.9	mass
a_1	competition coefficient of coral on algae	0.005	1/(mass.time)
a_2	maximal growth rate	0.2	1/time
a_3	half saturation constant	0.25	mass
b_1	growth efficiency	0.35	-
b_2	density dependent coefficient of Herbivorous	0.015	1/(mass.time)
d_1	the rate of decline	0.02	1/time
r_2	intrinsic growth rate of coral	0.6	1/time
k_2	carrying capacity of coral	0.7	mass
c_1	competition coefficient of algae on coral	0.8	1/(mass.time)

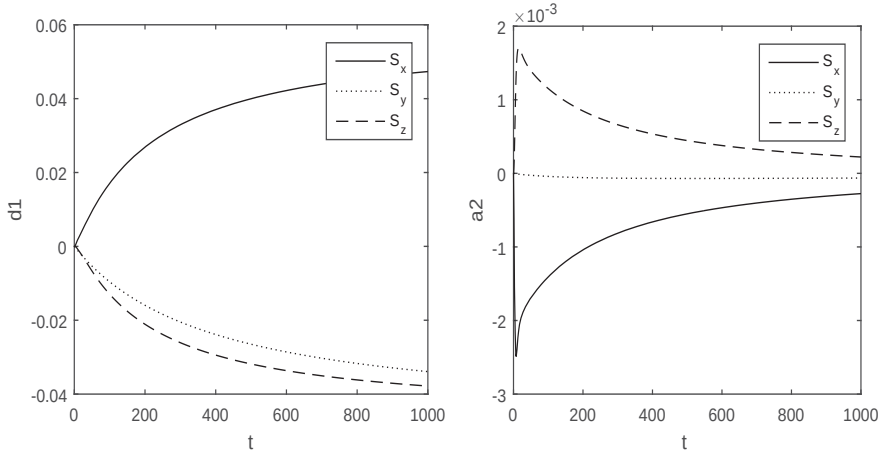


Figure 1: Time dependent sensitivity of the model (2.1) on Algae, Herbivores and Corals with respect to d_1 and a_2 .

6.1 Sensitivity Analysis

In this section we numerically check which parameters have more effects on our solution of system (2.1). Concerning the conservation of coral reefs we provide a graph for the changes in sensitivity index for each species with respect to parameters like a_2, d_1, a_1, b_2 and c_1 . First, the following values of parameters $a_2 = 0.0033$ and $d_1 = 0.0004$ have been used for constructing Fig.1, and similarly, Fig-2 has been constructed by using the values of parameters $a_1 = 0.2$, $b_2 = 0.007$ and $c_1 = 0.4$. Here, the growth of herbivores is declining with respect to the parameters b_2 and d_1 . In Fig-1 and Fig-2, it is also observed that the effect of these two parameters on our solution of system 2.1 are same and their positive effect on algae and negative effect on corals are also observed, where a_1 and c_1 are competition coefficient of corals and algae respectively and it appears as if corals and algae have been replaced by each other.

To show sensitivity index with respect to parameter a_2 , first we derivative the solution with respect to a_2 and define the sensitivity index (Arriola and Hyman, 2009).

$$S_{X_{a_2}} = \lim_{\delta a_2 \rightarrow 0} \left(\frac{\delta x}{x} \right) \left(\frac{\delta a_2}{a_2} \right)^{-1} = \frac{a_2}{x} \frac{\delta x}{\delta a_2}.$$

This normalized sensitivity index measures the relative change in the algae $\frac{\delta x}{x}$ with a relative small change to the competition parameter $\frac{\delta a_2}{a_2}$.

similar processes are going on for other parameters and species too. For competition coefficients a_1 and c_1 we have sensitivity index, where a very little effect on herbivores is observed for an increase in the value of competition coefficients, although competition coefficient of corals is more effective than competition coefficient of algae to the behavioral changes of herbivores. For an increase in the values of the maximal growth rate of herbivore parameter a_2 , the number of macro algae declines and the number of corals increases as herbivores feed on algae.

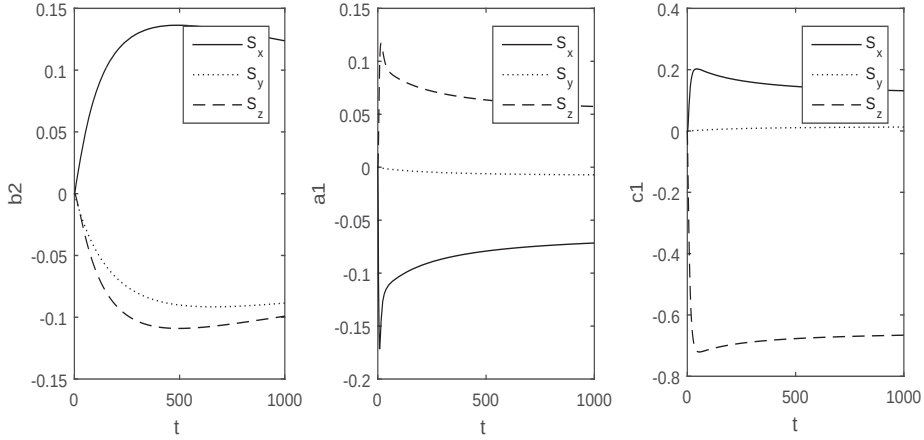


Figure 2: Time dependent sensitivity of the model (2.1) on Algae, Herbivores and Corals with respect to b_2 , a_1 and c_1 .

6.2 Time series Analysis

In this section we have described time series numerically. For the parameter values which are given in Table-1 we obtain a unique positive equilibrium $E^* = (0.1791, 1.1862, 0.5314)$ which satisfies the condition (3.4) that confirms the stability of interior point. Fig-3 clearly shows that a stability graph is obtained using the parameter values which are taken from Table-1. If the subsequent values of b_2 and d_1 from Table-1 are changed, then Fig-4 and Fig-5 are obtained respectively, i.e Fig-4 is obtained by using $b_2 = 0.014$ and it gives the eigen values $(-0.6997, -0.348, 1.199)$, similarly in Fig-5, we take $d_1 = 0.01$ and the eigenvalues $(-3.8734, 0.4066, 1.8189)$ are obtained. It proves that Fig-4 and Fig-5 representing the system (2.1) is unstable. From these figures, we can draw the conclusion that the system (2.1) loses its stability if the number of Herbivores decline. In fig-6 the impact of corals on algae, present in the system (2.1) is illustrated i.e we have increased the value of a_1 and compared the two graphs obtained by taking two different initial values $I_1 = (0.2, 0.3, 0.5)$, and $I_2 = (0.2, 12, 0.5)$.

6.3 Hopf-Bifurcation

In previous section, in Fig-3, we have seen that the system (2.1) is stable when we take the values of parameters from Table-1. Now we gradually decrease the value of b_2 and it is seen that the system (2.1) enters into a limit cycle oscillation from stable position, i.e the system (2.1) loses its stability for decreasing the value of b_2 and the system undergoes a Hopf-bifurcation at $b_2 = b_2^* \approx 0.0148$. Previously, we have seen that the transversality condition for the Hopf-bifurcation is $\frac{d[Re\lambda(b_2)]}{db_2}|_{b_2=b_2^*} \neq 0$. Here, we get $\frac{d[Re\lambda(b_2)]}{db_2}|_{b_2=b_2^*} = 0.0478 \neq 0$ which satisfies the transversality condition for the Hopf-bifurcation. The concept mentioned above has been represented by a bifurcation diagram (note Fig-7). Another Bifurcation diagram (see Fig-8) is also given with respect to the parameter d_1 .

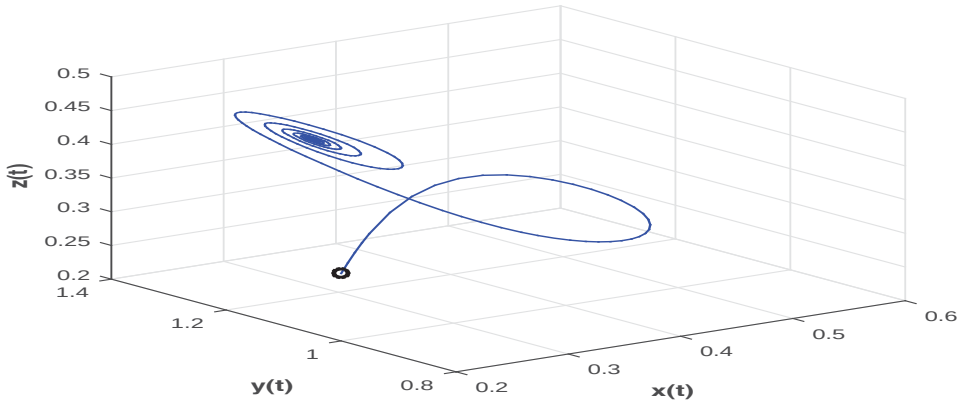


Figure 3: The system (2.1) gives a stable graph when the values of parameters are taken from table-1.

6.4 Competition

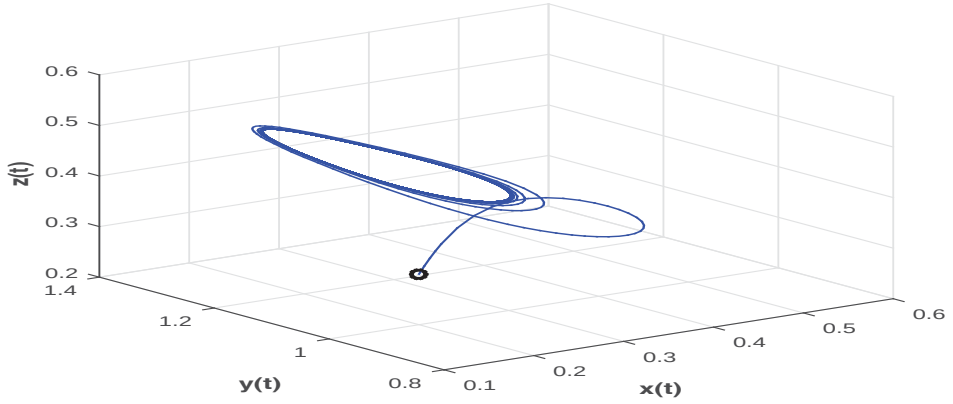
In this section, we illustrate the Graphical analysis of the two-species competitive equilibrium with the method of isoclines. Fig-9 shows a state occupied by algae only. It is observed that for any initial conditions, trajectories are reached at extreme isocline point and Fig-10 shows the state occupied by corals. Generally, the species with lower isocline becomes extinct when the intersection occurs outside the first quadrant. Fig-11 on the other hand shows the competitive coexistence, the intersection of the isoclines being a stable point. Here we have taken the parameter values such as $r_1 = 0.3, r_2 = 0.18, k_1 = 20, k_2 = 25$ and changing the value of competition coefficient we get the following graphs. We take the values $a_1 = 0.007, c_1 = 0.011$ and $a_1 = 0.012, c_1 = 0.006$ for Fig-9 and Fig-10 respectively.

7 Conclusions

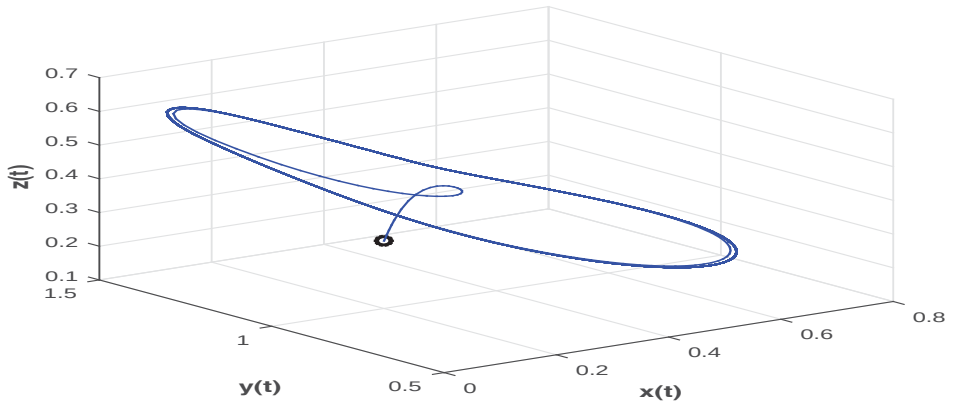
Considering the importance of activities of herbivorous fishes in a marine ecosystem the model 2.1 has been developed. For designing the model, in the beginning, natural death rate of herbivores was denoted by d_1 but if herbivorous fishes are harvested at constant harvesting rate in the system then the term $-hy$ will be added to the growth rate of herbivores. As a result, we would have obtained the term $-(d_1 + h)y$. However, skipping the procedure mentioned above, the numerical simulation of the model (2.1) was conducted over a wide range of values of d_1 to demonstrate harvesting effect of herbivorous fishes on the system.

We have considered three species i.e algae, herbivores and corals, where competition coefficient has been used to study the relation between corals and algae as the two species, which compete between them for oxygen and light. Moreover, a prey predator relationship has also been established between the algae and the herbivores.

Analytically, we have derived bifurcation with respect to b_2 and bifurcation curve has been es-



a



b

Figure 4: (a) The system (2.1) is unstable when $b_2 = 0.014$ and the values of other parameters are same as in table-1. (b) The system 2.1 is unstable when $d_1 = 0.01$ and the values of other parameters are unchanged.

established with respect to both b_2 and d_1 . From the observations mentioned above, it can be easily understood that herbivores play a critical role in determining the stability of this system (2.1), because the system gradually loses its stability with the decrease in number of herbivores. Apart from that, the impact of b_2 and d_1 on corals and algae has also been illustrated from sensitivity analysis.

Finally, we have demonstrated how the system gets affected in absence of herbivores. In conclusion, the study throws light on the competitive mechanism that exists between the corals and the algae and the coexistence of these two species in a coral reef ecosystem, which is considered to be playing a vital role in restoring the balance of the marine ecosystem.

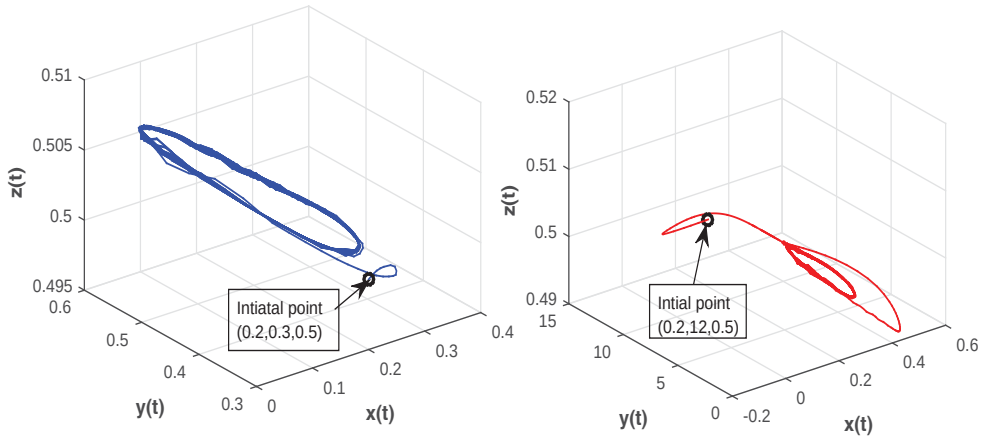


Figure 5: The system (2.1) is unstable when $r_1 = 0.8, a_1 = 0.75, r_2 = 0.35, c_1 = 0.03, k_2 = 0.51$

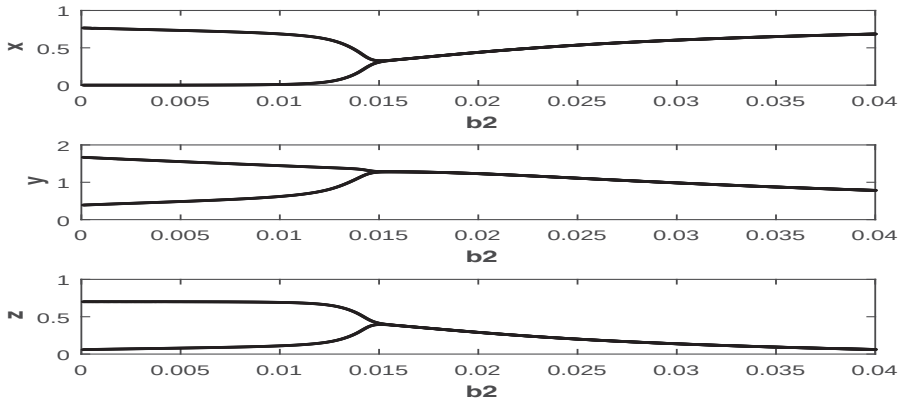


Figure 6: The system (2.1) goes through a bifurcation diagram with respect to the parameter b_2

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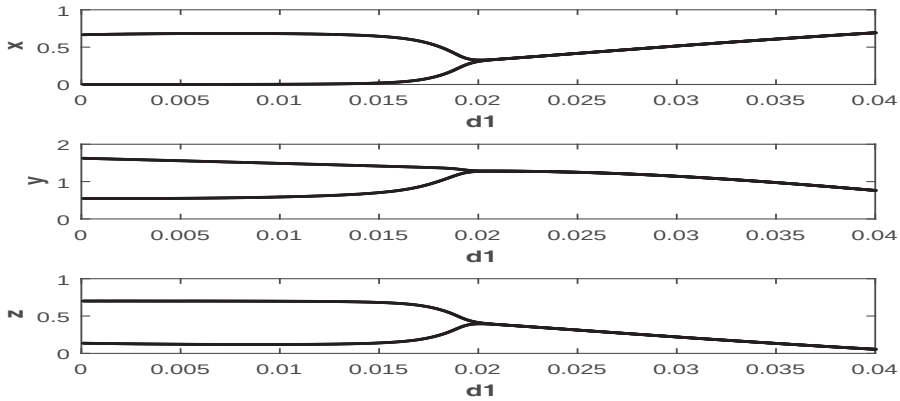


Figure 7: The system (2.1) goes through a bifurcation diagram with respect to the parameter d_1

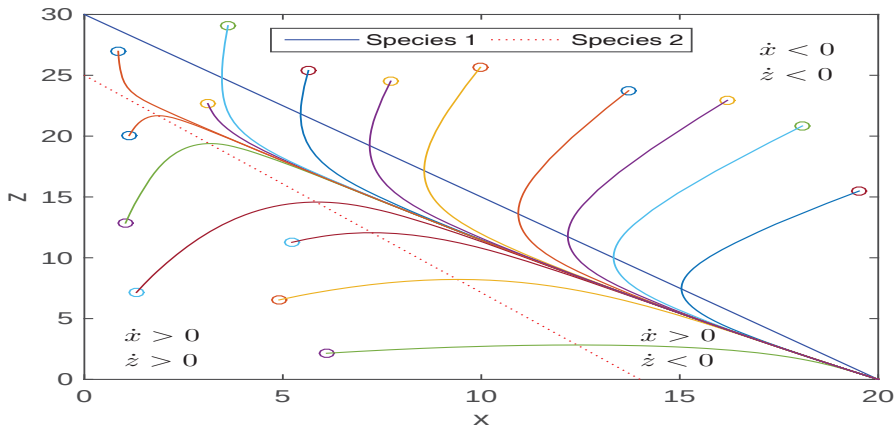


Figure 8: The trajectories of the system (5.1) with respect to different initial conditions represent a state occupied by algae only i.e a coral free state.

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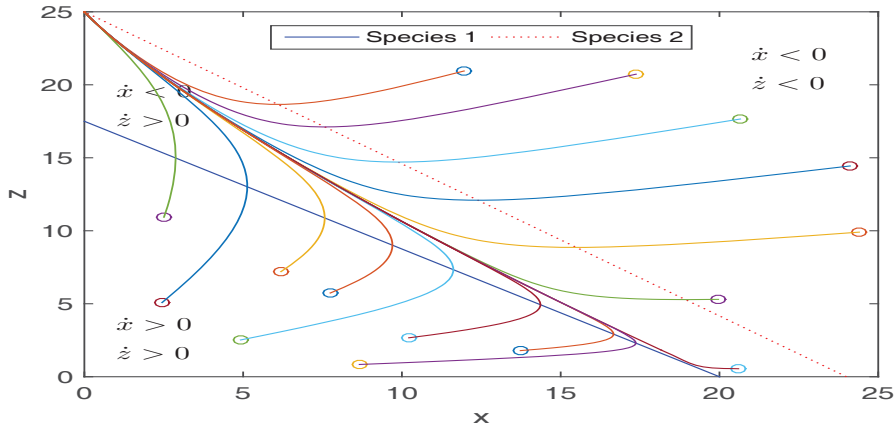


Figure 9: The trajectories of the system (5.1) with respect to different initial conditions represent a state occupied by corals only i.e an algae free state.

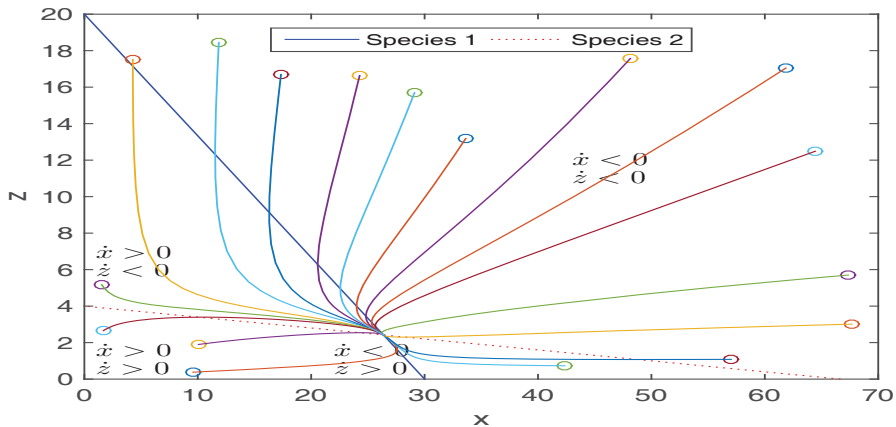


Figure 10: The system (5.1) shows coexistence as well as a stable graph with respect to different initial conditions.

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