

Control of Atmospheric Carbon Dioxide by Imposing Environmental Tax: A Mathematical Model

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ABSTRACT

The concentration of CO_2 is steadily rising from many sources such as from industrialization, deforestation etc. In this paper, a nonlinear mathematical model is proposed for the control of increasing CO_2 in the atmosphere. It is assumed that environmental tax imposed only when the concentration of CO_2 crosses a limit over which it becomes harmful for the environment. Criteria for local stability and instability of the non-negative equilibrium are obtained. We also obtained the condition for the global stability and the persistence of the system. Finally, we illustrated our results by numerical simulation.

Keywords: Mathematical Model, Equilibria and Stability, CO_2 , Industrialization, Environmental tax.

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1 Introduction

The gradual increase in the earth's temperature caused by high levels of carbon dioxide and other gases in the atmosphere is known as global warming (Chen and Li., 2011). Global warming is also referred as climate change. It has undergone more or less cyclic variations, some periods have been the warm climate and some cold. In many times, there was an abrupt change from one period to another. There are many factors which are responsible for these changes. Greenhouse gases are the one main cause of these changes (Mozumder, Randhir O.T. and Jerath, 2015). According to many climatologist, carbon-dioxide, methane, chlorofluorocarbons are the main greenhouse gases. Several studies shows that increase of greenhouse gases are the main cause of global warming. CO_2 is mainly responsible for global warming among other greenhouse gases. Due to the growing pressure of population, over industrialization and unplanned urbanization, the concentration of CO_2 is continuously increased into the environment. Excess of CO_2 traps heat and thus raising the temperature of the earth environment. The rise of atmospheric temperature results in sea level rise, impacts on agriculture, depletion of ozone layer, increased extreme weather, the spread of disease etc (Nikol'skii, 2010), (Woodwell, Hobbie J. E., Melillo J.M., Perterson and Shaver, 1983), (Ono, 2003)). Though CO_2 requires for the growth of the plants and it is also necessary to life on earth. It is also directly responsible for the food we eat and oxygen we

breathe but the increase of CO_2 in the environment has a harmful impact if it crosses its limit. Industrialization has always seemed to be the key to wealth and better living and it is also necessary for the economic growth of a country, but in reality, it affects the environment and ultimately contributes to climate change. Increasing industrialization is responsible for the increase in CO_2 into the environment. So there must be some strategies to control the concentration of CO_2 into the environment. In recent years, many policymakers proposed that environmental taxes including pollution charges are required to improve the environment. For example-carbon taxes, fuel duties etc. Carbon tax is a tax for controlling CO_2 into the atmosphere. It applied to the industries for every unit of carbon dioxide gas emission.

Many researchers have done several investigation to see the effect of increased CO_2 in our environment ((Sundar and Naresh, 2015),(Alexiadis, 2007),(Mccarty, 2001)). In particular,(Misra and Verma, 2013) have proposed a mathematical model to study the dynamics of carbon-dioxide gas in the atmosphere. (Misra and Verma, 2015) also studied a mathematical model for impact of environmental education on mitigation of carbon dioxide gas. (Agarwal and Devi, 2010) have studied the nonlinear mathematical model to study the concept of environmental tax and its effect on the survival of the biological population in the polluted environment. In this study, it is shown that cumulative rate of emission of pollutant is decreased due to the levy of taxes by assuming that pollutant from the external sources is dynamic in nature. (Sundar and Naresh, 2016) have proposed a mathematical model to study the effect of environmental tax on the carbon-dioxide emission. In this paper, it is assumed that environmental tax is imposed on the emitters only when the emission of carbon-dioxide gases crosses its limit. They have shown that environmental tax is an important factor to reduce global warming because the concentration of the carbon-dioxide gas reduces in the environment when environmental tax is imposed on the emitters.

Keeping all these papers in the mind, we have proposed a mathematical model to control the CO_2 into the environment. The main purpose of this paper is to control the concentration of carbon-dioxide gas into the atmosphere by introducing the concept of environmental tax. It has been assumed that environmental tax acts as a control parameter and applied only when the concentration of CO_2 crosses its limit over which it has a harmful impact in the environment.

2 Mathematical model

Let us consider any time t , $B(t)$, $I(t)$, $C(t)$ and $E(t)$ are the concentrations of biomass, Industrialization, carbon-dioxide and environmental tax respectively. In the view of above assumption we consider a mathematical model given by following set of differential equations:

$$\frac{dB}{dt} = sB - \frac{s_0 B^2}{L} - \alpha BI - \alpha_1 B^2 I + \pi \delta_1 BC, \quad (2.1)$$

$$\frac{dI}{dt} = \gamma I - \frac{\gamma_0 I^2}{K} + \alpha BI + \alpha_1 B^2 I, \quad (2.2)$$

$$\frac{dC}{dt} = Q(I) - \delta_0 C - \delta_1 BC - \mu E, \quad (2.3)$$

$$\frac{dE}{dt} = \nu(C - C_0) - \nu_0 E, \quad (2.4)$$

with the initial condition $B(0) \geq 0, I(0) \geq 0, C(0) > C_0, E(0) \geq 0$.

In the first equation, L is the carrying capacity of plant biomass. α is the depletion rate coefficient of plant biomass due to increase in industrialization. π is the rate at which CO_2 is used by the plant biomass during the photosynthesis process. In the second equation, K is the carrying capacity of industries. In the third equation, $Q(I)$, δ_0 is the emission rate and the natural depletion rate of CO_2 respectively. δ_1 is the depletion rate of CO_2 due to plant biomass. Reduce in CO_2 due to environmental tax is assumed to be at the rate μ . Finally, In the last equation ν is the tax rate coefficient and ν_0 is taken due to some practical difficulties on applying the reliable tax system. C_0 is the threshold concentration of CO_2 . Here we have taken three cases. In first case, emission rate of CO_2 is constant and for the second case emission rate of CO_2 is taken to be function of I . In the last, emission rate of CO_2 is assumed as a periodic function of I .

Now we discuss our model given by equation (2.1) to (2.4) for the following three cases:

- i) $Q(I) = Q_0 = \text{Constant}$.
- ii) $Q(I) = Q_0 + Q_1 I = \text{Linear function of } I$.
- iii) $Q(I)$ is periodic i.e. $Q(t) = Q_0 + \chi\psi(t), \chi(t + \omega) = \chi(t)$.
- i) For $Q(I) = Q_0$

3 Boundedness of the system

The boundedness of the system is obtained by following lemma:

Lemma 3.1. *All the solution of the system given by the equation (2.1) to (2.4) is bounded within the region:*

$$\phi = \{(B, I, C, E) : 0 < B + I + C + E \leq \frac{A}{\delta}\},$$

Provided $\pi < 1, \delta_0 > \nu + \delta$ and $\mu + \nu_0 > \delta$, where $A = Q_0 + \frac{(s + \delta)^2}{4\delta_0} + \frac{K(\gamma + \delta)^2}{4\gamma_0}$.

Proof:

$$\begin{aligned} \frac{dW}{dt} + \delta W &\leq Q_0 + sB - \frac{s_0 B^2}{L} + \gamma I - \frac{\gamma_0 I^2}{K} + (\nu - \delta_0)C - (\mu + \nu_0)E + \delta_1 BC(\pi - 1) + \delta W, \\ &\leq Q_0 + (s + \delta)B - \frac{s_0 B^2}{L} + (\gamma + \delta)I - \frac{\gamma_0 I^2}{K} - (\delta_0 - \nu - \delta)C - (\mu + \nu_0 - \delta)E, \\ &\leq Q_0 + (s + \delta)B - \frac{s_0 B^2}{L} + (\gamma + \delta)I - \frac{\gamma_0 I^2}{K}, \\ &\leq Q_0 + \frac{(s + \delta)^2}{4\delta_0} + \frac{K(\gamma + \delta)^2}{4\gamma_0}, \\ \frac{dW}{dt} &\leq A - \delta W, \end{aligned}$$

then by the usual comparison theorem as $t \rightarrow \infty$:

$$W_{\max} = \frac{A}{\delta},$$

Provided $\pi < 1, \delta_0 > \nu + \delta$ and $\mu + \nu_0 > \delta$.

This complete the proof of lemma (3.1).

4 Equilibrium Analysis

The model has four non-negative equilibrium points namely, $E_0(0, 0, \acute{C}, \acute{E})$, $E_1(0, \bar{I}, \bar{C}, \bar{E})$, $E_2(\breve{B}, 0, \breve{\acute{C}}, \breve{\acute{E}})$, $E_3(\hat{B}, \hat{I}, \hat{C}, \hat{E})$.

The value of the equilibrium points are given by solving the following equation:

$$\begin{aligned} sB - \frac{s_0 B^2}{L} - \alpha BI - \alpha_1 B^2 I + \pi \delta_1 BC &= 0, \\ \gamma I - \frac{\gamma_0 I^2}{K} + \alpha BI + \alpha_1 B^2 I &= 0, \\ Q_0 - \delta_0 C - \delta_1 BC - \mu E &= 0, \\ \nu(C - C_0) - \nu_0 E &= 0. \end{aligned}$$

1) For the equilibrium point $E_0(0, 0, \acute{C}, \acute{E})$

$$\begin{aligned} \acute{C} &= \frac{\nu C_0 + \nu_0 \acute{E}}{\nu}, \\ \acute{E} &= \frac{\nu(Q_0 - \delta_0 C_0)}{\delta_0 \nu_0 + \mu \nu}. \end{aligned}$$

Hence the equilibrium point E_0 exist if $Q_0 > \delta_0 C_0$.

2) For the equilibrium point $E_1(0, \bar{I}, \bar{C}, \bar{E})$

$$\begin{aligned} \bar{I} &= \frac{K\gamma}{\gamma_0}, \\ \bar{E} &= \frac{\nu(\bar{C} - C_0)}{\nu_0}, \\ \bar{C} &= \frac{Q_0 \delta_0 + \mu \nu C_0}{\delta_0 \nu_0 + \mu \nu}. \end{aligned}$$

Hence the equilibrium point E_1 exist without any condition.

3) For the equilibrium point $E_2(\breve{B}, 0, \breve{\acute{C}}, \breve{\acute{E}})$

$$\breve{\acute{C}} = \frac{Q_0 \nu_0 + \mu \nu C_0}{\delta_0 \nu_0 + \mu \nu + \delta_1 \breve{B} \nu_0} = f_1(\breve{B}), \quad (4.1)$$

$$\breve{\acute{E}} = \frac{\nu(f_1(\breve{B}) - C_0)}{\nu_0} = f_2(\breve{B}), \quad (4.2)$$

now, from the equation (4.1) and (4.2) we found that $\breve{\acute{C}}$ and $\breve{\acute{E}}$ are the functions of $\breve{B}$ only. For the existence of E_2 , we define a function given as:

$$\begin{aligned} F(\breve{B}) &= s - \frac{s_0 \breve{B}}{L} + \pi \delta_1 f_1(\breve{B}), \\ F(0) &= s + \pi \delta_1 f_1(0) > 0, \\ F\left(\frac{A}{\delta}\right) &= s - \frac{s_0 A}{\delta L} + \pi \delta_1 f_1\left(\frac{A}{\delta}\right) < 0, \end{aligned}$$

provided

$$s + \pi\delta_1 f_1\left(\frac{A}{\delta}\right) < \frac{s_0 A}{\delta L}.$$

We found that $F(0) > 0$ and $F(\frac{A}{\delta}) < 0$, showing the existence of $\check{B}$ in the interval $0 < \check{B} < \frac{A}{\delta}$. $\dot{F}(\check{B}) < 0$ is the sufficient conditions for $\check{B}$ to be unique.

$$\dot{F}(\check{B}) = \frac{-s_0}{L} - \frac{\pi\delta_1^2\nu_0(Q_0\nu_0 + \mu\nu C_0)}{(\delta_0\nu_0 + \delta_1\check{B} + \mu\nu)^2} < 0.$$

4) Now for the equilibrium point $E_3(\hat{B}, \hat{I}, \hat{C}, \hat{E})$

$$\hat{I} = \frac{K(\gamma + \alpha\hat{B} + \alpha\hat{B}^2)}{\gamma_0} = g_1(\hat{B}), \quad (4.3)$$

$$\hat{C} = \frac{Q_0\nu_0 + \mu\nu C_0}{\delta_1\hat{B}\nu_0 + \mu\nu + \delta_0\nu_0} = g_2(\hat{B}), \quad (4.4)$$

$$\hat{E} = \frac{\nu(g_2(\hat{B}) - C_0)}{\nu_0} = g_3(\hat{B}). \quad (4.5)$$

Now, from the equation (4.3), (4.4) and (4.5) we found that $\hat{I}, \hat{C}$ and $\hat{E}$ are the functions of $\hat{B}$ only. For the existence of E_3 , we define a function given as:

$$G(\hat{B}) = s - \frac{s_0\hat{B}}{L} - \alpha g_1(\hat{B}) - \alpha_1\hat{B}g_1(\hat{B}) + \pi\delta_1g_2(\hat{B}),$$

$$G(0) = s - \alpha g_1(0) + \pi\delta_1g_2(0),$$

$$G(0) = s - \frac{\alpha K\gamma}{\gamma_0} + \pi\delta_1g_2(0) > 0,$$

provided

$$\gamma_0 s > \alpha K\gamma.$$

$$G\left(\frac{A}{\delta}\right) = s - \frac{s_0 A}{\delta L} - \alpha g_1\left(\frac{A}{\delta}\right) - \alpha_1 \frac{A}{\delta} g_1\left(\frac{A}{\delta}\right) + \pi\delta_1 g_2\left(\frac{A}{\delta}\right),$$

$$G\left(\frac{A}{\delta}\right) < 0,$$

provided

$$s + \pi\delta_1 g_2\left(\frac{A}{\delta}\right) < \frac{s_0 A}{\delta L} + \alpha g_1\left(\frac{A}{\delta}\right) + \alpha_1 \frac{A}{\delta} g_1\left(\frac{A}{\delta}\right).$$

We found that $G(0) > 0$ and $G(\frac{A}{\delta}) < 0$, showing the existence of $\hat{B}$ in the interval $0 < \hat{B} < \frac{A}{\delta}$. $\dot{G}(\hat{B}) < 0$ is the sufficient conditions for $\hat{B}$ to be unique.

$$G'(\hat{B}) = -\frac{s_0}{L} - (\alpha + \alpha_1\hat{B})g_1'(\hat{B}) + \pi\delta_1g_2'(\hat{B}),$$

$$G'(\hat{B}) < 0,$$

Provided

$$g_2'(\hat{B}) < 0.$$

5 Stability Analysis

To investigate local stability of the system around all the equilibrium points, the variational matrix V is given as:

$$V = \begin{pmatrix} s - \frac{2s_0B}{L} - \alpha I - 2\alpha_1BI + \pi\delta_1C & -\alpha B - \alpha_1B^2 & \pi\delta_1B & 0 \\ \alpha I + 2\alpha_1BI & \gamma - \frac{2\gamma_0I}{K} + \alpha B + \alpha_1B^2 & 0 & 0 \\ -\delta_1C & 0 & -\delta_0 - \delta_1B & -\mu \\ 0 & 0 & \nu & -\nu_0 \end{pmatrix}$$

1) For the equilibrium point $E_0(0, 0, \dot{C}, \dot{E})$

$$V(E_0) = \begin{pmatrix} s + \pi\delta_1\dot{C} & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ -\delta_1\dot{C} & 0 & -\delta_0 & -\mu \\ 0 & 0 & \nu & -\nu_0 \end{pmatrix},$$

the variational matrix $V(E_0)$ has four eigenvalues in which two eigenvalues are $s + \pi\delta_1\dot{C} > 0$ and $\gamma > 0$, showing that this equilibrium point will be always unstable.

2) For the equilibrium point $E_1(0, \bar{I}, \bar{C}, \bar{E})$

$$V(E_1) = \begin{pmatrix} s - \alpha\bar{I} + \pi\delta_1\bar{C} & 0 & 0 & 0 \\ \alpha\bar{I} & \gamma - \frac{2\gamma_0\bar{I}}{K} & 0 & 0 \\ -\delta_1\bar{C} & 0 & -\delta_0 & -\mu \\ 0 & 0 & \nu & -\nu_0 \end{pmatrix},$$

the variational matrix $V(E_1)$ has four eigenvalues in which two eigenvalues are $s - \alpha\bar{I} + \pi\delta_1\bar{C}$ and $\gamma - \frac{2\gamma_0\bar{I}}{K}$, showing that this equilibrium point will be stable if $\alpha\bar{I} > \pi\delta_1\bar{C} + s$ and $\gamma < \frac{2\gamma_0\bar{I}}{K}$. The equilibrium point is unstable if either one or both of the conditions are not satisfied.

3) For the equilibrium point $E_2(\check{B}, 0, \check{C}, \check{E})$,

$$V(E_2) = \begin{pmatrix} s - \frac{2s_0\check{B}}{L} + \pi\delta_1\check{C} & -\alpha\check{B} - \alpha_1\check{B}^2 & \pi\delta_1\check{B} & 0 \\ 0 & \gamma + \alpha\check{B} + \alpha_1\check{B}^2 & 0 & 0 \\ -\delta_1\check{C} & Q_1 & -\delta_0 - \delta_1\check{B}^2 & -\mu \\ 0 & 0 & \nu & -\nu_0 \end{pmatrix},$$

The variational matrix $V(E_2)$ also has four eigenvalues in which one eigenvalue is $\gamma + \alpha\check{B} + \alpha_1\check{B}^2 > 0$, showing that this equilibrium point will be always unstable.

4) Now, For the equilibrium point $E_3(\hat{B}, \hat{I}, \hat{C}, \hat{E})$

In the following theorem we obtained the sufficient conditions for E_3 to be locally stable.

Theorem 5.1. *The interior equilibrium point $E_3(\hat{B}, \hat{I}, \hat{C}, \hat{E})$ is nonlinearly asymptotically locally stable within the region of attraction ϕ without any condition*

Proof. To prove the theorem, we first linearize the model by taking

$$B = \hat{B} + B_1, I = \hat{I} + I_1, C = \hat{C} + C_1, E = \hat{E} + E_1,$$

where B_1, I_1, C_1, E_1 are the small perturbation around the equilibrium point $E_3(\hat{B}, \hat{I}, \hat{C}, \hat{E})$

After linearization, the model is:

$$\frac{dB_1}{dt} = \frac{-s_0 \hat{B} B_1}{L} - \alpha_1 \hat{B} \hat{I} B_1 - \alpha \hat{B} I_1 - \alpha_1 \hat{B}^2 I_1 + \pi \delta_1 \hat{B} C_1, \quad (5.1)$$

$$\frac{dI_1}{dt} = \frac{-\gamma_0 \hat{I} I_1}{K} + \alpha B_1 \hat{I} + 2\alpha_1 \hat{B} \hat{I} B_1, \quad (5.2)$$

$$\frac{dC_1}{dt} = -\delta_0 C_1 - \delta_1 \hat{B} C_1 - \delta_1 B_1 \hat{C} - \mu E_1, \quad (5.3)$$

$$\frac{dE_1}{dt} = \nu C_1 - \nu_0 E_1. \quad (5.4)$$

Now we consider a following positive definite function

$$U = \frac{1}{2} \frac{B_1^2}{\hat{B}} + \frac{1}{2} \frac{K_1 I_1^2}{\hat{I}} + \frac{1}{2} K_2 C_1^2 + \frac{1}{2} K_3 E_1^2, \quad (5.5)$$

where K_1, K_2, K_3 are positive constants taken to be suitably.

Differentiating U with respect to time t , we get,

$$\frac{dU}{dt} = \frac{B_1}{\hat{B}} \frac{dB_1}{dt} + \frac{K_1 I_1}{\hat{I}} \frac{dI_1}{dt} + K_2 C_1 \frac{dC_1}{dt} + K_3 E_1 \frac{dE_1}{dt},$$

After putting the values of the derivatives from the equations (12.1)-(12.4) we get,

$$\begin{aligned} \frac{dU}{dt} = & -B_1^2 \left(\frac{s_0}{L} + \alpha_1 \hat{I} \right) + B_1 I_1 (-\alpha - \alpha_1 \hat{B} + \alpha K_1 + 2\alpha_1 \hat{B} K_1) + B_1 C_1 (\pi \delta_1 - \delta_1 \hat{C} K_2) \\ & - \frac{K_1 \gamma_0 I_1^2}{K} - C_1^2 (\delta_0 K_2 + \delta_1 \hat{B} K_2) + C_1 E_1 (\nu K_3 - \mu K_2) - \nu_0 K_3 E_1^2, \end{aligned}$$

now choosing,

$K_1 = \frac{\alpha + \alpha_1 \hat{B}}{\alpha + 2\alpha_1 \hat{B}}, K_2 = \frac{\pi}{\hat{C}}$ and $K_3 = \frac{\mu \pi}{\nu \hat{C}}$, we found that $\frac{dU}{dt}$ will be negative definite within the region defined by ϕ without any condition.

Hence the theorem is proved. $\square$

Theorem 5.2. Let the following inequalities holds:

$$\left(\frac{\alpha_1 A}{\delta} \right)^2 < \frac{2s_0 \gamma_0}{LK}, \quad (5.6)$$

$$(\pi \delta_1)^2 < \frac{s_0}{L} (\delta_0 + \delta_1 \hat{B}), \quad (5.7)$$

$$(\nu - \mu)^2 < 2\nu_0 (\delta_0 + \delta_1 \hat{B}), \quad (5.8)$$

then E_3 is globally asymptotically stable.

Proof. Let us consider following positive definite function

$$V = \left(B - \hat{B} - \hat{B} \log \frac{B}{\hat{B}} \right) + \left(I - \hat{I} - \hat{I} \log \frac{I}{\hat{I}} \right) + \frac{1}{2} (C - \hat{C})^2 + \frac{1}{2} (E - \hat{E})^2,$$

now differentiating V with respect to time t we get,

$$\frac{dV}{dt} = \left(\frac{B - \hat{B}}{B} \right) \frac{dB}{dt} + \left(\frac{I - \hat{I}}{I} \right) \frac{dI}{dt} + (C - \hat{C}) \frac{dC}{dt} + (E - \hat{E}) \frac{dE}{dt},$$

after putting the value of $\frac{dB}{dt}$, $\frac{dI}{dt}$, $\frac{dC}{dt}$, $\frac{dE}{dt}$ from the equation (2.1) to (2.4) we get,

$$\begin{aligned} \frac{dV}{dt} = & -\frac{1}{2}a_{11}(B - \hat{B})^2 + a_{12}(B - \hat{B})(I - \hat{I}) - a_{22}(I - \hat{I})^2 - \frac{1}{2}a_{11}(B - \hat{B})^2 + a_{13}(B - \hat{B})(C - \hat{C}) \\ & - \frac{1}{2}a_{33}(C - \hat{C})^2 - \frac{1}{3}a_{33}(C - \hat{C})^2 + a_{34}(C - \hat{C})(E - \hat{E}) - a_{44}(E - \hat{E})^2, \end{aligned}$$

where

$$\begin{aligned} a_{11} &= \frac{s_0}{L} + \alpha_1 I, a_{12} = \alpha_1 B, a_{22} = \frac{\gamma_0}{K}, \\ a_{13} &= \pi \delta_1 - \delta_1 C, a_{33} = \delta_0 + \delta_1 \hat{B}, \\ a_{34} &= (\nu - \mu), a_{44} = \nu_0. \end{aligned}$$

Then the sufficient condition for $\frac{dV}{dt}$ to be negative definite are:

$$\begin{aligned} a_{12}^2 &< 2a_{11}a_{22}, \\ a_{13}^2 &< a_{11}a_{33}, \\ a_{34}^2 &< 2a_{33}a_{44}. \end{aligned}$$

After maximizing L.H.S and minimizing the R.H.S. of the above equations, we found that conditions for the global stability are similar to the conditions given by equations (5.6) to (5.8). Hence theorem is proved. $\square$

6 Persistence of the System

Theorem 6.1. *If the conditions $\delta s > \alpha A$ and $\delta Q_0 > \mu A$ are satisfied then system of equations (2.1) to (2.4) is persistence.*

Proof. Let us suppose that $(\hat{B}, \hat{I}, \hat{C}, \hat{E})$ be the unique positive solution of the system of equation from (2.1) to (2.4),

From the equation (2.1)

$$\frac{dB}{dt} \geq B(s - \alpha I_{\max}) - B^2 \left(\frac{s_0}{L} + \alpha_1 I_{\max} \right),$$

after applying the usual comparison theorem as $t \rightarrow \infty$ we get:

$$B \geq \frac{L(\delta s - \alpha A)}{(\delta s_0 + L\alpha_1 A)}.$$

From equation (2.2) we get:

$$\frac{dI}{dt} \geq \gamma I - \frac{\gamma_0 I^2}{K},$$

after applying the usual comparison theorem as $t \rightarrow \infty$ we get:

$$I \geq \frac{K\gamma}{\gamma_0}.$$

From equation (2.3) we get:

$$\frac{dC}{dt} \geq Q_0 - \delta_0 C - \delta_1 B_{\max} C - \mu E_{\max},$$

after applying the usual comparison theorem as $t \rightarrow \infty$ we get:

$$C \geq \frac{\delta Q_0 - \mu A}{\delta_0 \delta + \delta_1 A}.$$

From equation (2.4) we get:

$$\frac{dE}{dt} \geq \nu(C_{\min} - C_0) - \nu_0 E,$$

after applying the usual comparison theorem as $t \rightarrow \infty$ we get:

$$E \geq \frac{\nu(C_{\min} - C_0)}{\nu_0}.$$

Hence theorem is proved. □

ii) For $Q(I) = Q_0 + Q_1 I = \text{Linear function of } I$.

7 Boundedness of the system

The boundedness of the system is obtained by following lemma:

Lemma 7.1. *All the solution of the system given by the equation (2.1) to (2.4) is bounded within the region:*

$$\psi = \{(B, I, C, E) : 0 < B + I + C + E \leq \frac{A_1}{\eta}\},$$

provided $\pi < 1, \delta_0 > \nu + \delta$ and $\mu + \nu_0 > \delta$, where $A_1 = Q_0 + \frac{(s + \delta)^2}{4\delta_0} + \frac{K(\gamma + Q_1 + \eta)^2}{4\gamma_0}$.

$$\begin{aligned}
 \frac{dW}{dt} + \delta W &\leq Q_0 + sB - \frac{s_0 B^2}{L} + (\gamma + Q_1)I - \frac{\gamma_0 I^2}{K} + (\nu - \delta_0)C - (\mu + \nu_0)E + \delta_1 BC(\pi - 1) + \delta W, \\
 &\leq Q_0 + (s + \delta)B - \frac{s_0 B^2}{L} + (\gamma + Q_1 + \delta)I - \frac{\gamma_0 I^2}{K} - (\delta_0 - \nu - \delta)C - (\mu + \nu_0 - \delta)E, \\
 &\leq Q_0 + (s + \delta)B - \frac{s_0 B^2}{L} + (\gamma + Q_1 + \delta)I - \frac{\gamma_0 I^2}{K}, \\
 &\leq Q_0 + \frac{(s + \delta)^2}{4\delta_0} + \frac{K(\gamma + Q_1 + \delta)^2}{4\gamma_0}, \\
 \frac{dW}{dt} &\leq A_1 - \delta W,
 \end{aligned}$$

Then by the usual comparison theorem as $t \rightarrow \infty$:

$$W_{\max} = \frac{A_1}{\delta},$$

provided $\pi < 1, \delta_0 > \nu + \delta$ and $\mu + \nu_0 > \delta$.

8 Equilibrium Analysis

The model has four non-negative equilibrium points namely $E_{00}(0, 0, \dot{C}_1, \dot{E}_1)$, $E_{11}(0, \bar{I}_1, \bar{C}_1, \bar{E}_1)$, $E_{22}(\check{B}_1, 0, \check{C}_1, \check{E})$, $E_{33}(\hat{B}_1, \hat{I}_1, \hat{C}_1, \hat{E}_1)$.

The value of the equilibrium points are given by solving the following equations:

$$\begin{aligned}
 sB - \frac{s_0 B^2}{L} - \alpha BI - \alpha_1 B^2 I + \pi \delta_1 BC &= 0, \\
 \gamma I - \frac{\gamma_0 I^2}{K} + \alpha BI + \alpha_1 B^2 I &= 0, \\
 Q(I) - \delta_0 C - \delta_1 BC - \mu E &= 0, \\
 \nu(C - C_0) - \nu_0 E &= 0.
 \end{aligned}$$

The value of the equilibrium points $E_{00}(0, 0, \dot{C}_1, \dot{E}_1)$, and $E_{22}(\check{B}_1, 0, \check{C}_1, \check{E}_1)$ will be same as $E_0(0, 0, \dot{C}, \dot{E})$, $E_2(\check{B}, 0, \check{C}, \check{E})$ respectively as I is zero.

1) For the equilibrium point $E_{11}(0, \bar{I}_1, \bar{C}_1, \bar{E}_1)$,

$$\begin{aligned}
 \bar{I}_1 &= \frac{K\gamma}{\gamma_0}, \\
 \bar{E}_1 &= \frac{\nu(\bar{C}_1 - C_0)}{\nu_0}, \\
 \bar{C}_1 &= \frac{Q_0\delta_0 + Q_1K\gamma + \mu\nu C_0}{\delta_0\nu_0 + \mu\nu}.
 \end{aligned}$$

2) For the equilibrium point $E_{33}(\hat{B}_1, \hat{I}_1, \hat{C}_1, \hat{E}_1)$

$$\hat{I}_1 = \frac{K(\gamma + \alpha\hat{B}_1 + \alpha\hat{B}_1^2)}{\gamma_0} = r_1(\hat{B}_1), \quad (8.1)$$

$$\hat{C}_1 = \frac{Q_0\nu_0 + Q_1\nu_0r_1(\hat{B}) + \mu\nu C_0}{\delta_1\hat{B}_1\nu_0 + \mu\nu + \delta_0\nu_0} = r_2(\hat{B}_1), \quad (8.2)$$

$$\hat{E}_1 = \frac{\nu(r_2(\hat{B}_1) - C_0)}{\nu_0} = r_3(\hat{B}_1). \quad (8.3)$$

From the equation (8.1) , (8.2) and (8.3), we found that $\hat{I}_1$, $\hat{C}_1$ and $\hat{E}_1$ are the functions of $\hat{B}_1$ only. For the existence of E_{33} , we define a function given as:

$$\begin{aligned} R(\hat{B}_1) &= s - \frac{s_0\hat{B}_1}{L} - \alpha r_1(\hat{B}_1) - \alpha_1\hat{B}_1r_1(\hat{B}_1) + \pi\delta_1r_2(\hat{B}_1), \\ R(0) &= s - \alpha r_1(0) + \pi\delta_1r_2(0), \\ R(0) &= s - \frac{\alpha K\gamma}{\gamma_0} + \pi\delta_1r_2(0) > 0, \end{aligned}$$

provided

$$\gamma_0 s > \alpha K \gamma.$$

$$R\left(\frac{A_1}{\delta}\right) = s - \frac{s_0 A_1}{\delta L} - \alpha r_1\left(\frac{A_1}{\delta}\right) - \alpha_1 \frac{A_1}{\delta} r_1\left(\frac{A_1}{\delta}\right) + \pi \delta_1 r_2\left(\frac{A_1}{\delta}\right),$$

$$R\left(\frac{A_1}{\delta}\right) < 0,$$

provided

$$s + \pi\delta_1r_2\left(\frac{A_1}{\delta}\right) < \frac{s_0 A_1}{\delta L} + \alpha r_1\left(\frac{A_1}{\delta}\right) + \alpha_1 \frac{A_1}{\delta} r_1\left(\frac{A_1}{\delta}\right).$$

We found that $R(0) > 0$ and $R\left(\frac{A_1}{\delta}\right) < 0$, showing the existence of $\hat{B}_1$ in the interval $0 < \hat{B}_1 < \frac{A_1}{\delta}$. $\dot{G}(\hat{B}_1) < 0$ is the sufficient conditions for $\hat{B}_1$ to be unique.

$$\begin{aligned} R'(\hat{B}_1) &= -\frac{s_0}{L} - (\alpha + \alpha_1\hat{B}_1)r_1'(\hat{B}) + \pi\delta_1r_2'(\hat{B}_1) \\ R'(\hat{B}_1) &< 0, \end{aligned}$$

Provided

$$r_2'(\hat{B}) < 0.$$

9 Stability Analysis

To investigate local stability of the system around all the equilibrium points, the variational matrix M is given as:

$$M = \begin{pmatrix} s - \frac{2s_0B}{L} - \alpha I - 2\alpha_1BI + \pi\delta_1C & -\alpha B - \alpha_1B^2 & \pi\delta_1B & 0 \\ \alpha I + 2\alpha_1BI & \gamma - \frac{2\gamma_0I}{K} + \alpha B + \alpha_1B^2 & 0 & 0 \\ -\delta_1C & Q_1 & -\delta_0 - \delta_1B & -\mu \\ 0 & 0 & \nu & -\nu_0 \end{pmatrix}$$

1) For the equilibrium point $E_{00}(0, 0, \dot{C}_1, \dot{E}_1)$

$$M(E_{00}) = \begin{pmatrix} s + \pi\delta_1\dot{C}_1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ -\delta_1\dot{C}_1 & 0 & -\delta_0 & -\mu \\ 0 & 0 & \nu & -\nu_0 \end{pmatrix},$$

the variational matrix $M(E_{00})$ has four eigenvalues in which two eigenvalues are $s + \pi\delta_1\dot{C}_1 > 0$ and $\gamma > 0$, showing that this equilibrium point will be always unstable.

2) For the equilibrium point $E_{11}(0, \bar{I}_1, \bar{C}_1, \bar{E}_1)$

$$M(E_{11}) = \begin{pmatrix} s - \alpha\bar{I}_1 + \pi\delta_1\bar{C}_1 & 0 & 0 & 0 \\ \alpha\bar{I}_1 & \gamma - \frac{2\gamma_0\bar{I}_1}{K} & 0 & 0 \\ -\delta_1\bar{C}_1 & 0 & -\delta_0 & -\mu \\ 0 & 0 & \nu & -\nu_0 \end{pmatrix},$$

the variational matrix $M(E_{11})$ has four eigenvalues in which two eigenvalues are $s - \alpha\bar{I}_1 + \pi\delta_1\bar{C}_1$ and $\gamma - \frac{2\gamma_0\bar{I}_1}{K}$, showing that this equilibrium point will be stable if $\alpha\bar{I}_1 > \pi\delta_1\bar{C}_1 + s$ and $\gamma < \frac{2\gamma_0\bar{I}_1}{K}$. Equilibrium point will be unstable if either one or both of the conditions are not satisfied.

3) For the equilibrium point $M_{22}(\check{B}_1, 0, \check{C}_1, \check{E}_1)$

$$M(E_{22}) = \begin{pmatrix} s - \frac{2s_0\check{B}_1}{L} + \pi\delta_1\check{C}_1 & -\alpha\check{B}_1 - \alpha_1\check{B}_1^2 & \pi\delta_1\check{B}_1 & 0 \\ 0 & \gamma + \alpha\check{B}_1 + \alpha_1\check{B}_1^2 & 0 & 0 \\ -\delta_1\check{C}_1 & Q_1 & -\delta_0 - \delta_1\check{B}_1^2 & -\mu \\ 0 & 0 & \nu & -\nu_0 \end{pmatrix},$$

the variational matrix $M(E_{22})$ also has four eigenvalues in which one eigenvalue is $\gamma + \alpha\check{B}_1 + \alpha_1\check{B}_1^2 > 0$ showing that, this equilibrium point will be always unstable.

4) For the equilibrium point $M_{33}(\hat{B}_1, \hat{I}_1, \hat{C}_1, \hat{E}_1)$,

In the following theorem we obtained the sufficient conditions for M_{33} to be locally stable.

Theorem 9.1. *let the following inequality hold:*

$$\frac{Q_1^2\pi^2}{\hat{C}_1^2} < \frac{4\gamma_0(\alpha + \alpha_1\hat{B}_1)(\delta_0 + \delta_1\hat{B}_1)}{K(\alpha + 2\alpha_1\hat{B}_1)}, \quad (9.1)$$

then M_{33} is locally asymptotically stable.

Proof. To prove the theorem, we first linearize the model by taking

$$B = \hat{B}_1 + B_{11}, I = \hat{I}_1 + I_{11}, C = \hat{C}_1 + C_{11}, E = \hat{E}_1 + E_{11},$$

where $B_{11}, I_{11}, C_{11}, E_{11}$ are the small perturbation around the equilibrium point $E_{33}(\hat{B}_1, \hat{I}_1, \hat{C}_1, \hat{E}_1)$. After linearization the model is:

$$\frac{dB_{11}}{dt} = \frac{-s_0 \hat{B}_1 B_{11}}{L} - \alpha_1 \hat{B}_1 \hat{I}_1 B_{11} - \alpha \hat{B}_1 I_{11} - \alpha_1 \hat{B}_1^2 I_{11} + \pi \delta_1 \hat{B}_1 C_{11}, \quad (9.2)$$

$$\frac{dI_{11}}{dt} = \frac{-\gamma_0 \hat{I}_1 I_{11}}{K} + \alpha B_{11} \hat{I}_1 + 2\alpha_1 \hat{B}_1 \hat{I}_1 B_{11}, \quad (9.3)$$

$$\frac{dC_{11}}{dt} = Q_1 I_{11} - \delta_0 C_{11} - \delta_1 \hat{B}_1 C_{11} - \delta_1 B_{11} \hat{C}_1 - \mu E_{11}, \quad (9.4)$$

$$\frac{dE_{11}}{dt} = \nu C_{11} - \nu_0 E_{11}. \quad (9.5)$$

Now we consider a following positive definite function

$$U_1 = \frac{1}{2} \frac{B_{11}^2}{\hat{B}_1} + \frac{1}{2} \frac{K_{11} I_{11}^2}{\hat{I}_1} + \frac{1}{2} K_{22} C_{11}^2 + \frac{1}{2} K_{33} E_{11}^2, \quad (9.6)$$

where K_{11}, K_{22}, K_{33} are positive constants taken to be suitably.

Differentiating U_1 with respect to time t , we get,

$$\frac{dU_1}{dt} = \frac{B_{11}}{\hat{B}_1} \frac{dB_{11}}{dt} + \frac{K_{11} I_{11}}{\hat{I}_1} \frac{dI_{11}}{dt} + K_{22} C_{11} \frac{dC_{11}}{dt} + K_{33} E_{11} \frac{dE_{11}}{dt},$$

after putting the values of the derivatives from the equations (12.6)-(12.9) we get,

$$\begin{aligned} \frac{dU_1}{dt} = & -B_{11}^2 \left(\frac{s_0}{L} + \alpha_1 \hat{I}_1 \right) + B_{11} I_{11} (-\alpha - \alpha_1 \hat{B}_1 + \alpha K_{11} + 2\alpha_1 \hat{B}_1 K_{11}) + B_{11} C_{11} (\pi \delta_1 - \delta_1 \hat{C}_1 K_{22}) \\ & - \frac{K_{11} \gamma_0 I_{11}^2}{K} + Q_1 K_{22} C_{11} I_{11} - C_{11}^2 (\delta_0 K_{22} + \delta_1 \hat{B}_1 K_{22}) + C_{11} E_{11} (\nu K_{33} - \mu K_{22}) - \nu_0 K_{33} E_{11}^2, \end{aligned}$$

now choosing,

$$K_{11} = \frac{\alpha + \alpha_1 \hat{B}_1}{\alpha + 2\alpha_1 \hat{B}_1}, K_{22} = \frac{\pi}{\hat{C}_1} \text{ and } K_{33} = \frac{\mu \pi}{\nu \hat{C}_1},$$

Then $\frac{dU_1}{dt}$ reduced in following form:

$$\frac{dU_1}{dt} = -B_{11}^2 \left(\frac{s_0}{L} + \alpha_1 \hat{I}_1 \right) - \frac{K_{11} \gamma_0 I_{11}^2}{K} + Q_1 K_{22} C_{11} I_{11} - C_{11}^2 (\delta_0 K_{22} + \delta_1 \hat{B}_1 K_{22}) - \frac{\nu_0 \pi \mu \hat{B}_1}{\nu \hat{C}_1} E_{11}^2,$$

from this we found that sufficient condition for $\frac{dU_1}{dt}$ to be negative definite is

$$\frac{Q_1^2 \pi}{\hat{C}_1} < \frac{4\gamma_0 (\alpha + \alpha_1 \hat{B}_1) (\delta_0 + \delta_1 \hat{B}_1)}{K (\alpha + 2\alpha_1 \hat{B}_1)}.$$

.Hence the theorem is proved. $\square$

Theorem 9.2. *Let the following inequalities holds:*

$$\left(\frac{\alpha_1 A_1}{\delta}\right)^2 < \frac{s_0 \gamma_0}{LK}, \quad (9.7)$$

$$(\pi \delta_1)^2 < \frac{2}{3} \frac{s_0}{L} (\delta_0 + \delta_1 \hat{B}_1), \quad (9.8)$$

$$Q_1^2 < \frac{2}{3} \frac{\gamma_0}{K} (\delta_0 + \delta_1 \hat{B}_1), \quad (9.9)$$

$$(\nu - \mu)^2 < \frac{4}{3} \nu_0 (\delta_0 + \delta_1 \hat{B}_1), \quad (9.10)$$

then E_{33} is globally asymptotically stable.

Proof. Let us consider following positive definite function,

$$V_1 = \left(B - \hat{B}_1 - \hat{B}_1 \log \frac{B}{\hat{B}_1}\right) + \left(I - \hat{I}_1 - \hat{I}_1 \log \frac{I}{\hat{I}_1}\right) + \frac{1}{2} (C - \hat{C}_1)^2 + \frac{1}{2} (E - \hat{E}_1)^2,$$

now differentiating V with respect to time t we get,

$$\frac{dV_1}{dt} = \left(\frac{B - \hat{B}_1}{B}\right) \frac{dB}{dt} + \left(\frac{I - \hat{I}_1}{I}\right) \frac{dI}{dt} + (C - \hat{C}_1) \frac{dC}{dt} + (E - \hat{E}_1) \frac{dE}{dt},$$

after putting the value of $\frac{dB}{dt}$, $\frac{dI}{dt}$, $\frac{dC}{dt}$, $\frac{dE}{dt}$ from the equation (2.1) to (2.4) we get,

$$\begin{aligned} \frac{dV_1}{dt} = & -\frac{1}{2} b_{11} (B - \hat{B}_1)^2 + b_{12} (B - \hat{B}_1) (I - \hat{I}_1) - \frac{1}{2} b_{22} (I - \hat{I}_1)^2 - \frac{1}{2} b_{11} (B - \hat{B}_1)^2 + b_{13} (B - \hat{B}_1) (C - \hat{C}_1) \\ & - \frac{1}{3} b_{33} (C - \hat{C}_1)^2 - \frac{1}{3} b_{33} (C - \hat{C}_1)^2 + b_{23} (C - \hat{C}_1) (I - \hat{I}_1) - \frac{1}{2} b_{22} (I - \hat{I}_1)^2 - \frac{1}{3} b_{33} (C - \hat{C}_1)^2 \\ & + b_{34} (C - \hat{C}_1) (E - \hat{E}_1) - b_{44} (E - \hat{E}_1)^2, \end{aligned}$$

where

$$\begin{aligned} b_{11} &= \frac{s_0}{L} + \alpha_1 B, b_{12} = \alpha_1 B, b_{22} = \frac{\gamma_0}{K}, \\ b_{13} &= \pi \delta_1 - \delta_1 C, b_{33} = \delta_0 + \delta_1 \hat{B}_1, \\ b_{23} &= Q_1, b_{34} = (\nu - \mu), b_{44} = \nu_0. \end{aligned}$$

Then the sufficient condition for $\frac{dV_1}{dt}$ to be negative definite are:

$$\begin{aligned} b_{12}^2 &< b_{11} b_{22}, \\ b_{13}^2 &< \frac{2}{3} b_{11} b_{33}, \\ b_{23}^2 &< \frac{2}{3} b_{22} b_{33}, \\ b_{34}^2 &< \frac{4}{3} b_{33} b_{44}. \end{aligned}$$

After maximizing the L.H.S and minimizing the R.H.S. of the above equations, we found that conditions for the global stability of the system are similar to the conditions given by equations (9.7) to (9.10). Hence the theorem is proved. □

10 Persistence of the System

The condition of the persistence will be same as given in case i ($Q(I) = Q_0$). Case 3: $Q(t) = Q_0 + \chi\psi(t)$, $\psi(t + \omega) = \psi(t)$

In this case the model given by equation (2.1) – (2.4) can be written in the vector matrix form as:

$$\dot{x}(t) = A(x) + \chi F(t), x(0) = x_0 \quad (10.1)$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} B \\ I \\ C \\ E \end{pmatrix}, F(t) = \begin{pmatrix} 0 \\ 0 \\ \psi(t) \\ 0 \end{pmatrix}, x(0) = \begin{pmatrix} B(0) \\ I(0) \\ C(0) \\ E(0) \end{pmatrix},$$

$$A(x) = \begin{pmatrix} sx_1 - \frac{s_0 x_1^2}{L} - \alpha_1 x_1 x_2 - \alpha_1 x_1^2 x_2 + \pi \delta_1 x_1 x_3 \\ \gamma x_2 - \frac{\gamma_0 x_2^2}{K} + \alpha_1 x_1 x_2 + \alpha_1 x_1^2 x_2 \\ Q(x_2) - \delta_0 x_3 - \delta_1 x_1 x_3 - \mu x_4 \\ \nu(x_3 - C_0) - \nu_0 x_4 \end{pmatrix}.$$

Let us consider that $x(t, \sigma, \chi)$ be the solution of the equation (10.1) in such a manner that

$$x(0, \sigma, \chi) = \hat{x} + \sigma,$$

where

$$\hat{x} = \begin{pmatrix} \hat{B} \\ \hat{I} \\ \hat{C} \\ \hat{E} \end{pmatrix}, \sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{pmatrix},$$

here $x(t, 0, 0) = \hat{x}$.

We are interested to show the existence of periodic solution of period ω of the system given by equation (10.1) for sufficiently small χ , which tends to $\hat{x}$ as $\chi \rightarrow 0$.

For this we have to show that

$$x(\omega, \sigma, \chi) = \hat{x} + \sigma,$$

where σ is the function of χ for χ small, $\sigma \rightarrow 0$ as $\chi \rightarrow 0$.

Let us construct a function

$$\xi(\sigma, \chi) = x(\omega, \sigma, \chi) - \hat{x} - \chi$$

The existence of periodic solution can be prove by showing $\xi(\sigma, \chi) = 0$, has a solution for σ as as function of χ , $\sigma(\chi)$, such that $\sigma(0) = 0$. Now by using the implicit function (Freedman,1969).

$$\xi(0, 0) = x(\omega, 0, 0) - \hat{x} = \hat{x} - \hat{x} = 0$$

Then

$$\det \xi_{\sigma}(0, 0) = x_{\sigma}(\omega, 0, 0) - I$$

where I is the identity matrix. Now by applying the variational formula,

$$\dot{x}_{\sigma}(t, \sigma, \chi) = A_x(x(t, \sigma, \chi))x_{\sigma}(t, \sigma, \chi), x_{\sigma}(0, \sigma, \chi) = I,$$

from this we have, $x_{\sigma}(t, 0, 0)$ is the matrix solution of

$$\dot{x}_{\sigma}(t, 0, 0) = A_x(x(t, 0, 0))x_{\sigma}(t, 0, 0), x_{\sigma}(0, 0, 0) = I$$

But $x(t, 0, 0) = \hat{x}$ and $A_x(\hat{x}) = V(\hat{E}_1)$.

Thus,

$$\dot{x}_{\sigma}(t, 0, 0) = V(\hat{E}_1)x_{\sigma}(t, 0, 0), x_{\sigma}(0, 0, 0) = I.$$

After solving this, we obtained

$$\dot{x}_{\sigma}(t, 0, 0) = e^{V(\hat{E}_1)t}$$

So we have

$$\xi_{\sigma}(0, 0) = e^{V(\hat{E}_1)\omega} - I$$

Hence we obtained the following theorems.

Theorem 10.1. *If $V(\hat{E}_1)$ has no eigenvalues with zero real parts, then the system given by equations (2.1) – (2.4) with $Q(t) = Q_0 + \chi\psi(t)$, $\psi(t + \omega) = \psi(t)$ has a periodic solution of period ω , $(B(t, \chi), I(t, \chi), C(t, \chi), U(t, \chi), E(t, \chi))$ such that $(B(t, 0), I(t, 0), C(t, 0), U(t, 0), E(t, 0)) = (\hat{B}, \hat{I}, \hat{C}, \hat{U}, \hat{E})$.*

Theorem 10.2. *If $V(\hat{E}_1)$ has no eigenvalues with zero real parts, then for χ sufficiently small, the stability of the periodic solution is same as the stability of $\hat{E}_1$.*

Now we consider the periodic solution upto order of χ . For this we suppose that $x_{\chi}(t, \sigma, \chi)$ is the solution of

$$\dot{x}_{\chi}(t, \sigma, \chi) = A_x(x(t, \sigma, \chi))x_{\chi}(t, \sigma, \chi) + F(t), x_{\chi}(0, \sigma, \chi) = 0,$$

$x_{\chi}(t, 0, 0)$ can be found by the solution of

$$\dot{x}_{\chi}(t, 0, 0) = V(\hat{E}_1)x_{\chi}(t, 0, 0) + F(t), x_{\chi}(0, 0, 0) = 0,$$

Hence we have, $\dot{x}_{\chi}(t, 0, 0) = e^{V(\hat{E}_1)t} \int_0^t e^{-V(\hat{E}_1(s))} F(s) ds$.

Now applying the Taylor's series we have,

$$\begin{aligned} x_{\chi}(t, \sigma, \chi) &= \hat{x} + x_{\sigma}(t, 0, 0)\sigma + x_{\chi}(t, 0, 0)\chi + H.O.T. \\ \sigma(x) &= \sigma(0)\chi + o(\chi), \end{aligned}$$

where $\sigma(0) = -\xi_{\sigma}(0, 0)^{-1}\xi_{\chi}(0, 0)$ and $\xi_{\chi}(0, 0) = x_{\chi}(\omega, 0, 0)$. Thus we have,

$$\begin{aligned}\dot{\sigma}(0) &= -e^{-V(\hat{E}_1)} e^{-V(\hat{E}_1)} \int_0^\omega e^{-V(\hat{E}_1(s))} F(s) ds, \\ \dot{\sigma}(0) &= \int_0^\omega e^{-V(\hat{E}_1(s))} F(s) ds,\end{aligned}$$

which gives,

$$x_\chi(t, \sigma, \chi) = \hat{x} + \left[e^{V(\hat{E}_1(t))} \int_t^\omega e^{-V(\hat{E}_1(s))} F(s) ds \right] \chi + o(\chi).$$

This is the required periodic solution.

11 Numerical Analysis

Let us consider the following set of parameters

$$\begin{aligned}s &= 15, s_0 = 15, L = 100, \alpha = 0.0001, \alpha_1 = 0.00001, \pi_1 = 0.75, \delta_1 = 0.008, \delta = 2, \\ \gamma &= 10, \gamma_0 = 10, Q_0 = 40, \delta_0 = 3, \mu = 0.001, \nu = 0.05, \nu_0 = 2, C_0 = 10, K_1 = 60.\end{aligned}$$

For these values of the parameters the values of $E_3(\hat{B}, \hat{I}, \hat{C}, \hat{E})$ are given as :

$$\hat{B} = 99.9763, \hat{I} = 60.6597, \hat{C} = 10.5268, \hat{E} = .01317$$

The eigenvalues for the jacobian matrix corresponding to interior equilibrium point $E_3(\hat{B}, \hat{I}, \hat{C}, \hat{E})$ are $(-14.8335, -10.1129, -3.81895, -2.00003)$.

We found that all the eigenvalues are negative, hence the interior equilibrium point $E_3(\hat{B}, \hat{I}, \hat{C}, \hat{E})$ is locally asymptotically stable. For the case ii, when emission rate of CO_2 is the linear function of I : $Q_0=0.3$. Then the value of equilibrium point is $(100.167, 60.6621, 15.31, 0.13275)$ and the eigenvalues are $(-14.8335, -10.1129, -3.81895, -2.00003)$. Since all the eigenvalues are negative. Thus, the interior equilibrium point defined by $E_{33}(\hat{B}_1, \hat{I}_1, \hat{C}_1, \hat{E}_1)$ is also stable. The result of numerical simulation are displayed graphically. Figure(1) is a time series graph, In this figure, biomass, industries, concentration of CO_2 are plotted against time.

12 Conclusion

In this paper, A nonlinear mathematical model is proposed to control the concentration of CO_2 into the atmosphere. Here we have taken the concept of environmental tax, imposed only on the emitters when CO_2 crosses its limit beyond which it become harmful for the environment. Figures(2) and (3) are shown variation of concentration of CO_2 and environmental tax respectively for the different value of α_1 . From these figures it is found that as α_1 increases, the concentration of CO_2 and environmental tax also increases. In figures (4) and (5), the variation of concentration of CO_2 and environmental tax with time is shown respectively, for the different value of δ_0 (natural depletion rate of CO_2). From these figures, it is clear that as the natural depletion rate of CO_2 increases, concentration of CO_2 decreases, due to which environmental tax is also decreases. In the figures (6) and (7) the concentration of CO_2 and environmental tax are plotted with time for the different value of δ_1 (depletion rate of CO_2 due to biomass).

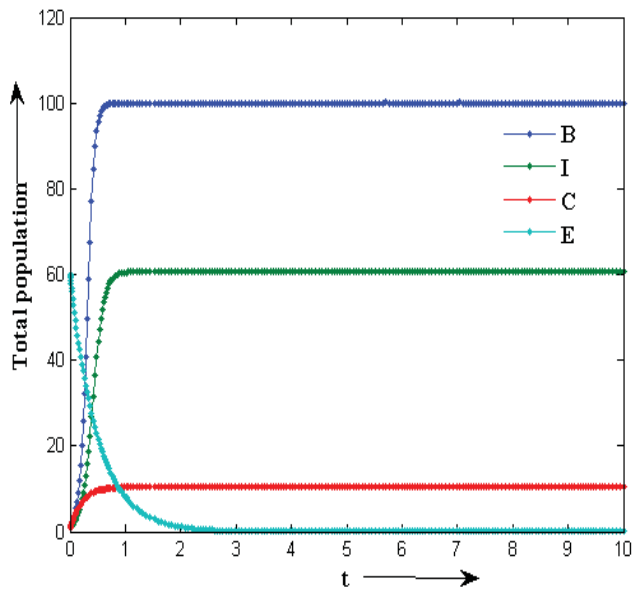


Figure 1: Time series graph

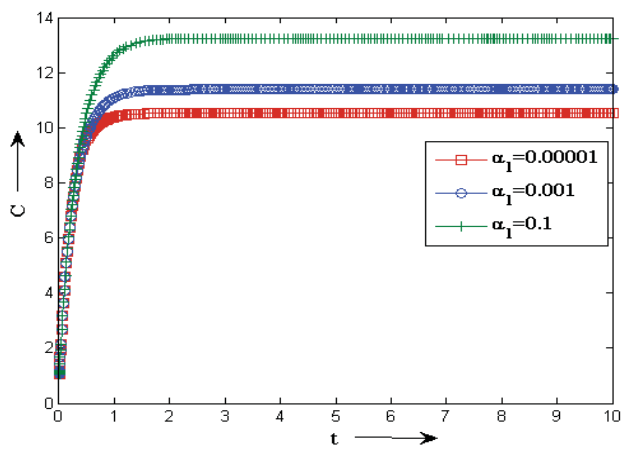


Figure 2: Variation of concentration of 'C' with time for different value of ' α_1 '

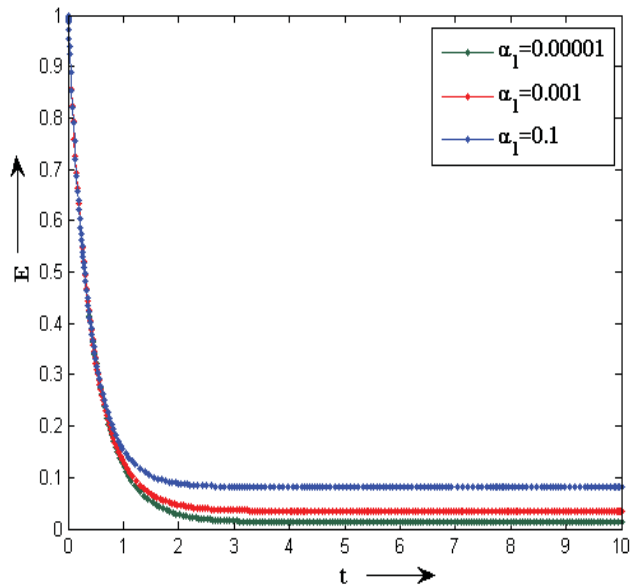


Figure 3: Variation of Environmental tax with time for different value of ' α_1 '

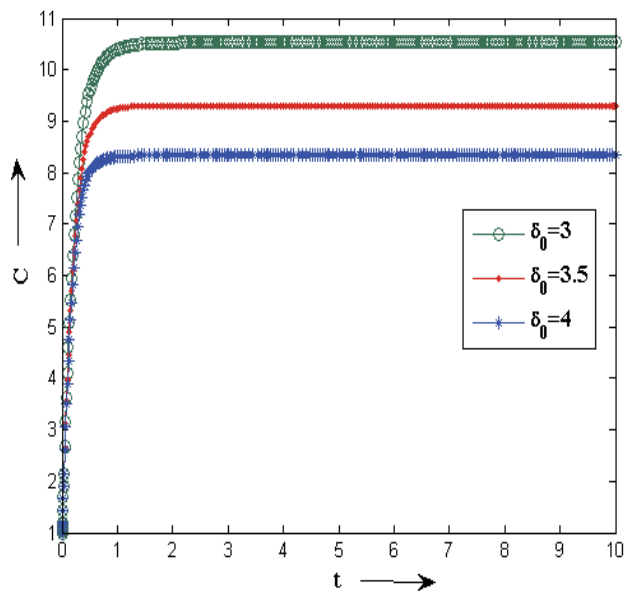


Figure 4: Variation of concentration of 'C' with time for different value of ' δ_0 '

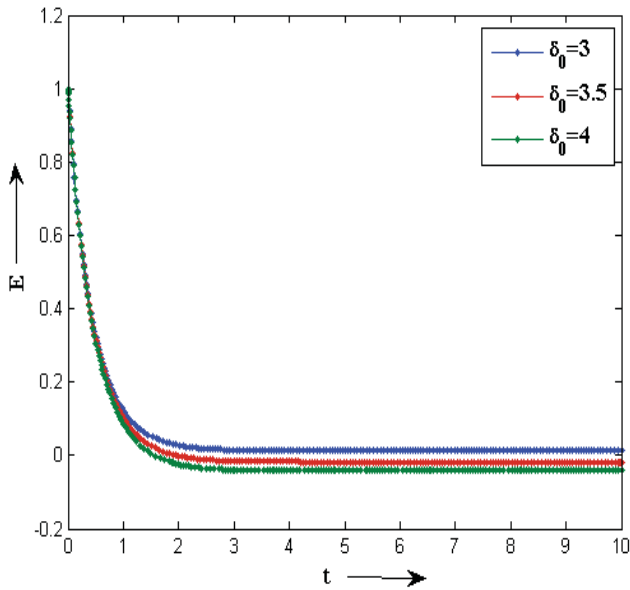


Figure 5: Variation of Environmental tax with time for different value of ' δ_0'

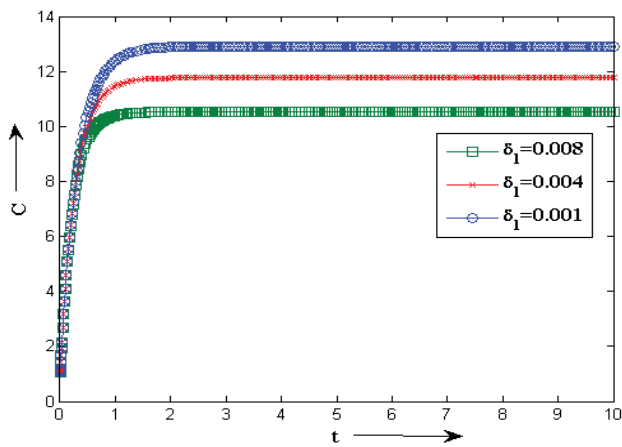


Figure 6: Variation of concentration of 'C' with time for different value of ' δ_1'

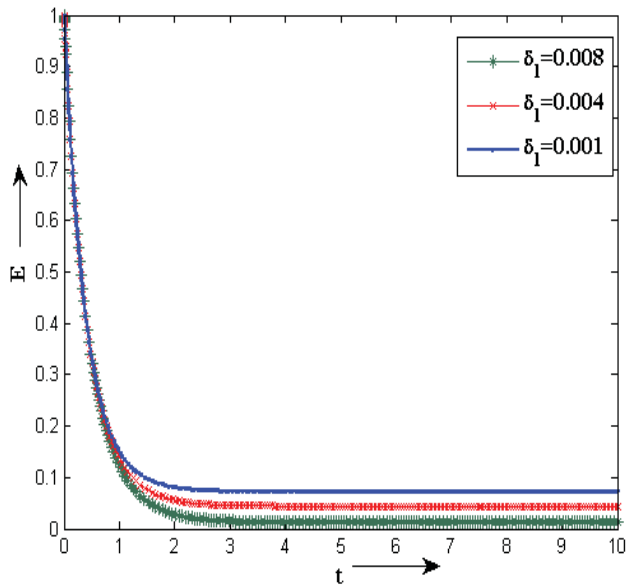


Figure 7: Variation of Environmental tax with time for different value of ' δ_1 '

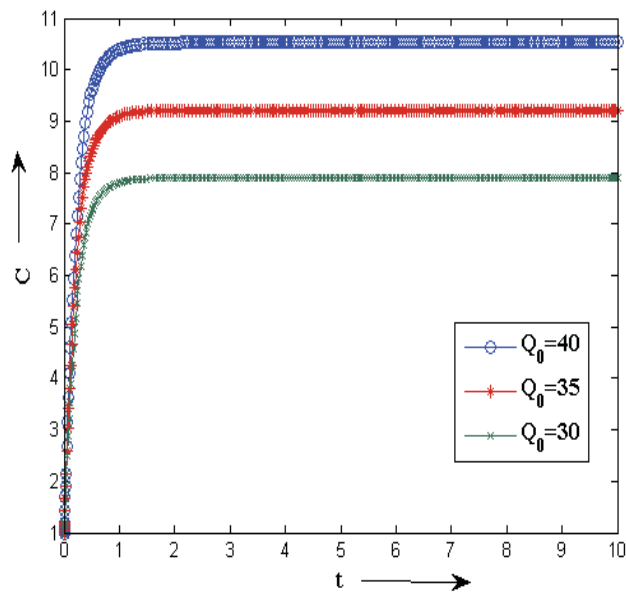


Figure 8: Variation of concentration of 'C' with time for different value of ' Q_0 '

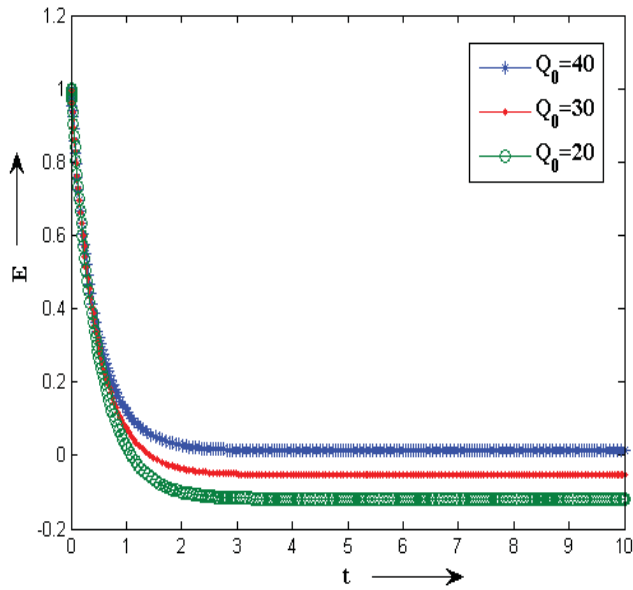


Figure 9: Variation of Environmental tax with time for different value of ' Q_0 '

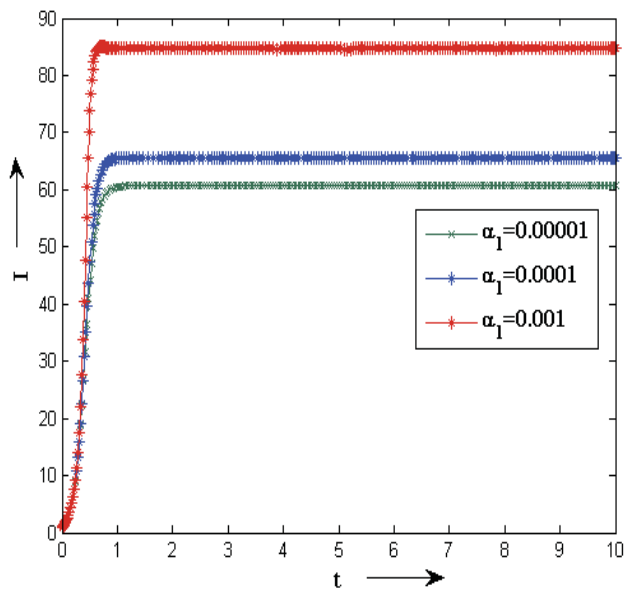


Figure 10: Variation of 'I' with time 't' for different value of ' α_1 '

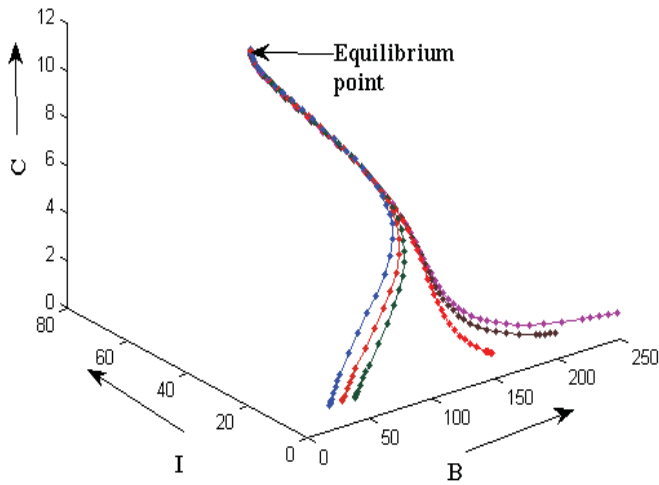


Figure 11: Graph between Biomass, Industries, concentration of 'C'

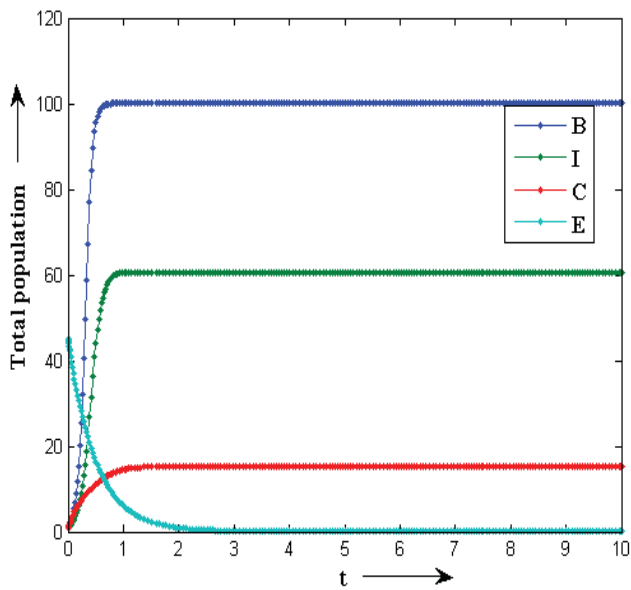


Figure 12: Time series graph

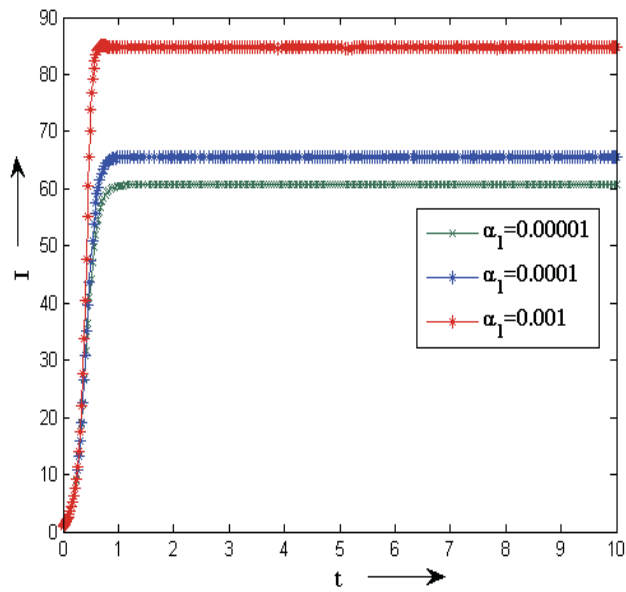


Figure 13: Variation of 'I' with time 't' for different value of ' α '

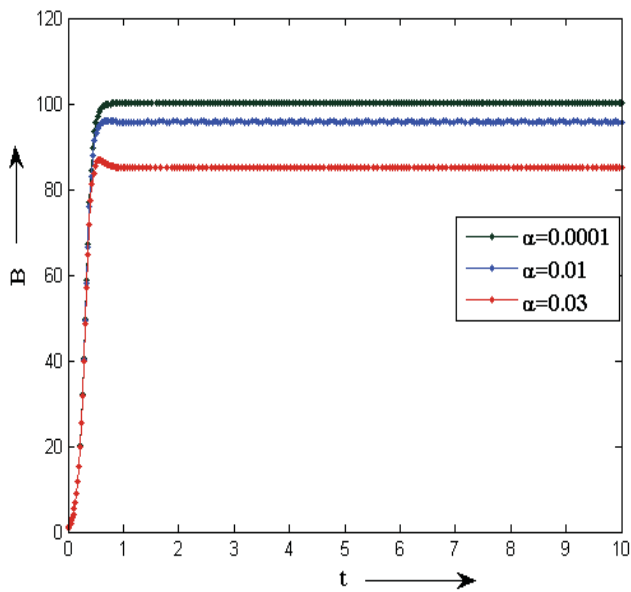


Figure 14: Variation of 'B' with time 't' for different value of ' α '

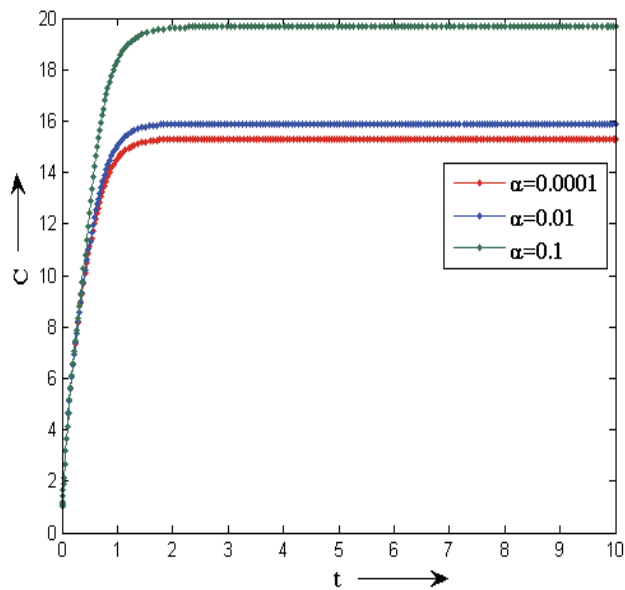


Figure 15: Variation of 'C' with time 't' for different value of α

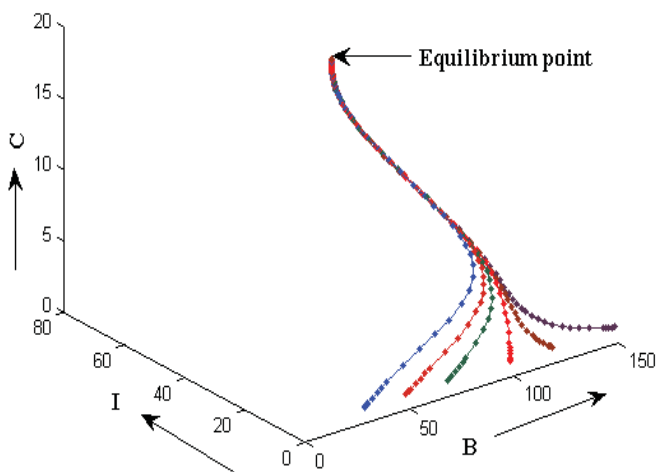


Figure 16: Graph between Biomass, Industries, concentration of 'C'

From these figures, we infer that as δ_1 increases, concentration of CO_2 and environmental tax is decreases. In figures (8) and (9), we show the variation of concentration of CO_2 and environmental tax with time for different value of Q_0 . When there is an increase in in this parameter, the value of Concentration of CO_2 and environmental tax is also increases. In figure (10), I is plotted with time for the different value of α_1 . Figure (11) is the graph between biomass, industries and concentration of CO_2 . In this figure, we obtained that the system given by equation(1-4) is globally stable as the trajectories initiating inside the region of attraction approaching towards the equilibrium point E_1 . Figures (13)-(17) are related to the second case, when emission rate of CO_2 is the function of I . Fig (13) is a time series graph. In figures (14),(15),(16), Industries, Biomass and concentration of CO_2 are plotted with time for different value of α . From these figures, we found that as α increase, biomass decrease but industries increase. Due to large numbers of industries concentration of CO_2 is also increases. Figure (17) is the graph of global stability between plant biomass, Concentration of CO_2 and industries. From the figure, we found that system is stable in B-I-C plane. So we have obtained that environmental tax is an important factor to reduce concentration of CO_2 from the atmosphere. Here we discussed three cases. In first case, when the emission rate of CO_2 is constant then system is locally stable without any condition but in the second case, when the emission rate of CO_2 is the linear function of I then system is locally stable under certain condition. We have also discussed the condition of global stability and persistence in both the cases. It has been pointed that a small periodic introduction of concentration of CO_2 into the atmosphere induces a periodic behavior in the system.

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