

An ecogenetic model with Michaelis-Menten functional response

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ABSTRACT

The present investigation deals with an eco-genetic model introducing Michaelis-Menten (ratio-dependent) functional response of two genetically distinguished prey population having influence of predator population on its evolution. A detailed mathematical analysis of the updated system are systematically described and find out the conditions for which no trajectory can reach the origin following any fixed direction or spirally. The predator-free sub system can be destabilized by the predators under a suitable parametric restriction. The model system reveals that the solution coexists at a stable steady state. Also conditions of non-existence of periodic solutions around coexistence equilibrium with the successful application of the Bendixson's criterion has been executed. Finally, a conclusion of the ecological implications of the analytical and numerical findings are discussed as well.

Keywords: An eco-genetic model; Ratio Dependent; Persistence; Global Stability ; Bifurcation; Ecological implications.

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1 Introduction

Mathematical epidemiology has become major field of study of research work for its own rights, since the seminal model of Kermack and McKendrick (1932) on SIRS (susceptible-infected-removed-susceptible) systems. The study of spreading disease on ecology in the development of predator-prey models have been thoroughly investigated from different feasible thinking to survive living thing in the ecosystem in last couple of decades [cf. Kermack and McKendrick (1932), Venturino (2001), Xiao and Chen (2001), Venturino (2002), Greenhalgh and Haque (2007), Rahman and Chakravarty (2013a), Rahman and Chakravarty (2013b), Haque et al. (2013)]. More recently epidemic models with dynamics of transmission and control of COVID-19 has been studied by many researchers [cf. Kucharski et al. (2020), Li et al. (2020), Tang et al. (2020)]. An upgrading in these above context it can be analyzed by Venturino (2012) and it has been observed that the disease is not a basic fundamental ingredient of the ecosystem, but it is fine to replace it by the insertion of two genotype applied to one of the population. After studying the eco-genetic modelling, I have seen that the author shares the notion with reference to genetically different characteristics population [cf. Venturino (2012), Viberti and Venturino (2014), Rahman (2016)] and determine the virtue of another population on its evolution.

Recently, it is interesting and more important to study the influence of the various types of the response function that recite the number of prey consumed by each predator per unit time. This function ordain the stability and bifurcation dynamics of the system.

The simplest and crude predator-prey response function is linear, also called Holling type I in which no handling time of the captured prey [cf. Korpimäki and Norrdahl (1991)]. Generally predators often perceive satiation, for this reason with abundant available food their consumption rate gradually decrease. By a linear response function is not enough to describe this phenomena. The predator-prey model with Holling type II response function has successfully been studied by [cf. Freedman (1980), May (2001), Molla et al. (2019), Haque et al. (2014)]. This illustrates a hyperbola having a horizontal asymptote, offers the large predation rate. Another innovation that can be used the Holling type IV response function, It pictures a humped curve which raises up from the origin and then diminishes at high prey densities, this reduce being occurred due to possible prey group defence. It is pointed out that although in Holling type II models predator perceive satiation, Holling type IV give better role in which the predation drops when the prey density is sufficiently large, due to for prey toxicity. The another most important functional response is Michaelis-Menton (ratio-dependent) response.

The classical prey-dependent predator-prey model point out not only the well known paradox of enrichment but also it had renamed as biological control paradox, which was firstly described by Luck [cf. Hairston et al. (1960), Rosenzweig (1971), Luck (1990)]. The detail study of the ratio-dependent predator-prey model express that it will generate neither the paradox of enrichment nor the biological control paradox [cf. Hsu et al. (2001b), Hsu et al. (2001a)]. It also permit mutual extinction of the eco-system [cf. Jost et al. (1999), Haque and Venturino (2007)]. Ratio-dependent models require high population densities for both prey and predator while the most interesting dynamics is around the origin [cf. Hsu et al. (2003), Arino et al. (2004), Haque and Venturino (2007)].

Some researchers formulate eco-genetic models influenced by Holling type II response function [cf. Venturino (2012), Viberti and Venturino (2014), Rahman (2016)]. It seems to me that more research work is needed in this field of area. So I propose a eco-genetic system introduced by Michaelis-Menton (ratio-dependent) functional response of two genetically distinguished prey population and one predator population which is new approach in this area. The main aim of this investigation is to uncover the effects of ratio dependent functional responses in a eco-genetic model. The dynamics of the origin starting from the first positive quadrant have analyzed and the probabilities of reaching this fixed point following any fixed direction in the domain of interest have been successfully studied. The reduction principle studied by Poincaré-Bendixson which allows us to reduce the system to a two-dimensional system. Also proper parametric restrictions has been derived by which the system cannot reach the origin spirally.

Based on the biological realistic point of view the mathematical model is formulated in the next section and Section 2 deals with the basic assumptions of the current model, mathematical analysis including steady states, existence, persistence, stability of the model, problem of finding some suitable restrictions by which there is no periodic solution around the positive equilibrium. In section 3, derive a sub-model to show the influence of predator in the system. In section 4 Simulations and Finally, section 5 concludes the paper with a brief discussion including biological implications of the key results of each section.

2 The Basic Model

2.1 System setup

Here we summarize the basic assumptions based on which the mathematical model is formulated.

(A1): It is assumed that there are two kinds of prey populations with genetically different characteristics. Let x and y denote two genotypes, that is, genetically differentiated prey population. Let r represents the reproduction rate of both prey, k is the carrying capacity of the prey population and p denote the fractions of the newborns being of genotypes x and $1 - p$ represent the same of the species y , respectively. Let us also assume that although they are genetically different but they live in a common ecological environment [cf. Venturino (2012), Viberti and Venturino (2014)].

(A2): The common predator species represented by z consume both of preys with ratio-dependent predation rate of [cf. Arino et al. (2004), since the Holling type II functional response is most popular among all the functional response used in last half century [cf. Arino et al. (2004), Haque and Venturino (2007)]. Let h and g be the predation rates of the two genotypes. e represents the conversion factor of prey into predators population, where $e < 1$.

(A3): We denote the population pressure on the prey of genotype x by a ; and b represents the same pressure on the prey of genotype y . The mortality rate of the predators is denoted by m .

With the above assumptions the model takes the following form

$$\frac{dx}{dt} = (rp - ax)(x + y) - \frac{hxz}{kz + x + y}, \quad (2.1a)$$

$$\frac{dy}{dt} = (r(1 - p) - by)(x + y) - \frac{gyz}{kz + x + y}, \quad (2.1b)$$

$$\frac{dz}{dt} = \frac{ez(hx + gy)}{kz + x + y} - mz. \quad (2.1c)$$

where we assume that all the system parameters are strictly positive. In the current study we restrict our attention to illustrated the dynamical behaviour of the eco-genetic model (2.1). The Table(1) reports the name of the variables and parameter values their units and the default values used in numerical simulations.

2.2 Boundedness

Here it is shown that the solution of the system (2.1) always exists and stays positive, that is the system is dissipative. also have the following results on the boundedness of solutions of system (2.1).

Proposition 2.2. The System (2.1) is dissipative.

Proof. From the first equation of the system (2.1), we can write

$$\begin{aligned} \frac{dx}{dt} &\geq (rp - ax)(x + y) \\ &\geq (rp - a(x + y))(x + y). \end{aligned} \quad (2.2)$$

In the similar way, From the second equation of the system (2.1), we can write similar way, From the second equation of the system (2.1), we can write

$$\frac{dy}{dt} \geq (r(1-p) - b(x+y))(x+y). \quad (2.3)$$

Therefore from 2.2 and 2.3 we get

$$\frac{d(x+y)}{dt} \geq (r - (a+b)(x+y))(x+y). \quad (2.4)$$

By using simple but standard argument we have

$$\limsup_{t \rightarrow +\infty} (x(t) + y(t)) \leq \frac{r}{a+b}.$$

Now there exists a $T_1 > 0$ such that for all $t > T_1$, we have

$$(x+y) < \frac{r}{a+b} + \epsilon_1 = W(say). \quad (2.5)$$

From the third equation of the system (2.1), we can write

$$\frac{dz}{dt} \leq \left(\frac{eW(h+g) - mkz}{kz + x + y} \right) z. \quad (2.6)$$

Therefore from 2.6 we get, by using simple but standard argument we have

$$\limsup_{t \rightarrow +\infty} (z(t)) \leq \frac{eW(h+g)}{mk}.$$

Now there exists a $T_2 > 0$ such that for all $t > T_2$, we have

$$z < \frac{eW(h+g)}{mk} + \epsilon_2 = W_1(say). \quad (2.7)$$

Which completes the proof.

Now it is concluded that for all $t > T$ such that $T = \max\{T_1, T_2\}$ then

$$(x+y) < W \quad \text{and} \quad (z) < W_1. \quad (2.8)$$

Theorem 2.2 All the solutions of system system (2.1) which initiate in R_+^3 are uniformly bounded.

Proof. We define a function

$$\tau = x + y + z \quad (2.2)(i)$$

The time derivative of the equation (2.2)(i) is given by

$$\frac{d\tau}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt}$$

for each $\mu > 0$, the following inequality holds,

$$\begin{aligned}
 \frac{d\tau}{dt} + \mu\tau &= (rp - ax)(x + y) - \frac{h x z}{kz + x + y} + (r(1 - p) - by)(x + y) \\
 &\quad - \frac{g y z}{kz + x + y} + \frac{e z (h x + g y)}{kz + x + y} - m z + \mu\tau \\
 &= (rx - ax^2 + \mu x) + (ry - ay^2 + \mu y) - (1 - e) \frac{z(hx + gy)}{kz + x + y} + (\mu - m)z, \\
 &\leq (rx - ax^2 + \mu x) + (ry - ay^2 + \mu y) \quad \text{assuming } m > \mu \text{ and since } e < 1 \\
 &\leq \frac{(r + \nu)^2(a + b)}{4ab} = \phi,
 \end{aligned}$$

Therefore, from the above inequation we have

$$\frac{d\tau}{dt} + \mu\tau \leq \phi.$$

Applying theory of differential inequality Birkhoff and Rota (1978), we obtain

$$0 < \tau(x, y, z) < \frac{\phi}{\mu}(1 - e^{-\mu t}) + \tau(x_0, y_0, z_0)e^{-\mu t}$$

and for $t \rightarrow \infty$ we have

$$0 < \tau < \frac{\phi}{\mu}.$$

Hence all solutions of system of equations (2.1) that initiate in R_+^3 are confined in the region

$$B = \left\{ (x, y, z) \in R_+^3 : \Omega = \frac{\phi}{\mu} + \epsilon \text{ for any } \epsilon > 0 \right\}.$$

2.2.1 Equilibria and their existence

The system (2.1) possesses three equilibrium points, namely (i) $E_0(x_0, y_0, z_0)$, (ii) $E_1(x_1, y_1, z_1)$, (iii) $E_2(x_2, y_2, z_2)$, where $x_0 = 0, y_0 = 0, z_0 = z_1 = 0, y_1 = \frac{r(1-p)}{b}, x_1 = \frac{rp}{a}, z_2 = \frac{(eh-m)x_2 - (eg-m)y_2}{mk}$ x_2 is the positive real root of the cubic equation in S ,

$$\theta_1 S^3 + \theta_2 S^2 + \theta_3 S + \theta_4 = 0 \quad (2.9)$$

where $\theta_1, \theta_2, \theta_3, \theta_4$ and y_2 are shown in the Appendix (A.1).

The equation (2.9) has exactly one real positive root if $G^2 + 4H^3 > 0$ where $G = \theta_1^2 \theta_4 - 3\theta_1 \theta_2 \theta_3 - 2\theta_2^3$ and $H = \theta_1 \theta_3 - \theta_2^2$. Using Cardon's method, we obtain the root as $\frac{1}{\theta_1}(Q - \frac{H}{Q} - \theta_2)$ where Q denotes one of the three values of $[\frac{1}{2}(-G + \sqrt{(G^2 + 4H^3)})^{\frac{1}{3}}]$. Here the equilibria E_0, E_1 are always exist. The coexistence equilibrium E_2 is exist which will be investigated numerically.

2.3 Stability analysis

The Jacobian matrix $J \equiv DF(s)$ of the system (2.1) at any arbitrary point (x, y, z) is given by $J(x, y, z) = (\delta_{ij})_{3 \times 3}$, where δ_{ij} 's are given by in the Appendix A.2.

2.3.1 The behaviour of the system around $E_0(0, 0, 0)$

Here the Jacobian matrix is undefined at the trivial equilibrium E_0 . So to analyze, procedure of Arino et al. (2004) is adopted here. Now (2.1) can be rewritten as

$$\frac{dX}{dt} = \Pi(X(t)) + \Omega(X(t)) \quad (2.10)$$

where $X = (x, y, z)^T$, $\Pi(X(t)) = (\Pi_1(X), \Pi_2(X), \Pi_3(X))^T$ is a continuously differentiable function homogenous of degree 1 and $\Omega = (\Omega_1(X), \Omega_2(X), \Omega_3(X))$ is another continuously differentiable function such that $\Omega(X) = O(X)$ at the origin. In this case $X = (x_1, x_2, x_3)^T = (x, y, z)^T$,

$$\begin{aligned} \Pi_1(X) &= rp(x_1 + x_2) - \frac{hx_1x_3}{kx_3 + x_1 + x_2}, & \Pi_2(X) &= r(1-p)(x_1 + x_2) - \frac{gx_2x_3}{kx_3 + x_1 + x_2} \\ \Pi_3(X) &= \frac{e(hx_1 + gx_2)x_3}{kx_3 + x_1 + x_2}, & \Omega_1(X) &= -ax_1(x_1 + x_2), \Omega_2(X) = -bx_2(x_1 + x_2), \Omega_3(X) = 0 \end{aligned}$$

Let the solution $X(t)$ is bounded and such that $\liminf_{t \rightarrow \infty} \|X(t)\| = 0$. From $\{X(t_n + s)\}_{t \geq 0}$, we can form a sequence $X(t_n + s)$, $t_n \rightarrow \infty$ such that $X(t_n + s) \rightarrow 0$ locally uniformly on $s \in R$. as $t_n \rightarrow \infty$. Define

$$y_n(s) = \frac{X(t_n + s)}{\|X(t_n + s)\|} \quad (2.11)$$

such that

$$\frac{dX(t_n + s)}{ds} = \Pi(X(t_n + s)) + \Omega(X(t_n + s)). \quad (2.12)$$

From $\Omega(X) = O(X)$, we can rewrite Ω as $\Omega(X) = \|X\|^2 O(1)$. Taking the derivative of (2.11), and after some calculation we have

$$\begin{aligned} \frac{dy_n}{ds} &= |\Pi(y_n(s)) - (y_n(s), \Pi(y_n(s)))y_n(s)| \\ &\quad + \|X(t_n + s)\|[\Omega(y_n(s)) - (y_n(s), \Omega(y_n(s)))y_n(s)] \end{aligned} \quad (2.13)$$

Clearly, y_n is bounded, $\|y_n(s)\| = 1$, $\forall s$, and $\frac{dy_n}{ds}$ is bounded too. So, applying the Ascoli-Arzelà theorem (see, Brezis et al. (1999)), one can extract from y_n a subsequence y_{n_u} locally uniformly converging to some function y for which as $t_{n_u} \rightarrow \infty$

$$\|X(t_n + s)\|[\Omega(y_n(s)) - (y_n(s), \Omega(y_n(s)))y_n(s)] \rightarrow 0.$$

and y satisfies the following system:

$$\frac{dy}{dt} = \Pi(y(t)) - (y(t), \Pi(y(t)))y(t), \quad \|y(t)\| = 1, \quad (2.14)$$

Equation (2.14) is defined for all $t \in R$.

Now the steady states of Π are given by the vectors α satisfying

$$\Pi(\alpha) = (\alpha, \Pi(\alpha))\alpha = \lambda\alpha \quad (2.15)$$

. called nonlinear eigenvalue problem. with $\|\alpha\| = 1$, These stationary solutions correspond to fixed directions that the trajectories of equation (2.14) may reach asymptotically. Equation (2.15) can be written as:

$$(\lambda - rp)\alpha_1^2 + (\lambda - 2rp)\alpha_1\alpha_2 + (\lambda k - rpk + h)\alpha_1\alpha_3 - rpk\alpha_3\alpha_2 - rp\alpha_2^2 = 0 \quad (2.16)$$

$$(\lambda - 2r(1-p))\alpha_1\alpha_2 + (\lambda - r(1-p))\alpha_2^2 + (\lambda k - r(1-p)k + g)\alpha_2\alpha_3 - rgk\alpha_1\alpha_3 - r(1-p)\alpha_1^2 = 0 \quad (2.17)$$

$$(\lambda + m - eh)\alpha_1\alpha_3 + (\lambda + m + eg)\alpha_2\alpha_3 + k(\lambda + m)\alpha_3^2 = 0 \quad (2.18)$$

Next I examine the possibility of reaching the origin following fixed directions.

Case I. $\alpha_1 = 0$

(i) $\alpha_2 = 0$ and $\alpha_3 \neq 0$ in this case, there is a possibility of reaching the origin following the Z-axis, with $\lambda = -m$.

(ii) $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$, in this case trajectories move away from the origin if $k > \max\{\frac{1}{e}, \frac{g}{r(1-p)}\}$ with positive eigenvalue $\frac{e(r(1-p)k-g)+m}{ek-1}$, on the other hand when $k > \max\{\frac{1}{e}, \frac{g}{rq}\}$ then trajectory can reach the origin.

(iii) $\alpha_2 \neq 0$ and $\alpha_3 = 0$ In this case trajectory would end on the Y axis with $\lambda = rp > 0$

Case II. $\alpha_2 = 0$ (i) $\alpha_1 \neq 0$ and $\alpha_3 = 0$ in this case, eigenvalue $\lambda = rp > 0$ So we cannot reach the origin through the X-axis; that is to say that the X-axis is not a fixed direction that the trajectories can follow to reach zero. (ii) $\alpha_1 \neq 0$ and $\alpha_3 \neq 0$ in this case, we have two possibilities (a) when $\frac{1}{e} < k < \frac{h}{rp}$ then there is no possibility of reaching the origin following the XZ-plane. (b) if $\frac{1}{e} > k > \frac{h}{rp}$ then trajectory can go the origin through the XZ-plane.

Case III. $\alpha_3 = 0$ and $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ in this case, the eigenvalue $\lambda = r$ is positive, then there is no possibility of reaching the origin following the XY-plane.

Case IV. $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$ In this case, the eigenvalues $\frac{h(h+e)}{2(k-1)}$ and $\frac{g(1+e)}{2(k-1)}$ are positive when $k > 1$, so there are two subcases (i) $k > 1$ In this case, there is no possibility of going to the origin following a fixed direction that is contained in the positive octant.

(ii) $k < 1$ Under this condition, the trajectories may follow a fixed direction that is contained in the positive octant.

Next, I try to find out some mathematical restriction by which no orbit of system (2.1) tends to the critical point spirally. Indeed, we have the following proposition.

Proposition 2.3.1(I). Let us assume $\Pi_i(X) = X_i \widetilde{\Pi}_i(X) \quad \forall i = 1, 2, 3$ then Equation (2.14) preserves positiveness. So Each feasible solution of (2.14) at a positive distance from at least one of the co-ordinate hyperplanes can be expound as a solution of an ordinary differential equation in R^2 , in that cases Poincare'-Bendixson criteria [cf. Hale and Koçak (2012)] can be applied.

Proof: Without loss of generality, we can assume that $y_3(t) \geq N > 0, \quad \forall t$ if y is non-trivial. Then $y_3 = \sqrt{1 - y_1^2 - y_2^2}$ and $\tilde{y} = (y_1, y_2)^T$ satisfies the system $\frac{d\tilde{y}(t)}{dt} = \tilde{\Theta}(\tilde{y}(t))$ with $\tilde{\Theta}$ being a

continuous function homogenous of degree one in an open subset of R_+^2 . Now as $t \rightarrow \infty$ either $\liminf y_i(t) = 0$ for all $i = 1, 2, 3$ or there is an index j for which for $t > 0$, in which case (2.14) can be reduced to its planar projection onto (y_i, y_k) , where $i, k \neq j$. Here take $j = 3$ without loss of generality; put then $\tilde{y} = (y_1, y_2)$ and $y_3 = \sqrt{1 - y_1^2 - y_2^2}$ and reduce the system to the following two-dimensional system

$$\frac{dy}{dt} = \tilde{\Pi}(\tilde{y}(t)) - \tilde{y}(t)|(\tilde{y}(t), \tilde{\Pi}(\tilde{y}(t))) + \sqrt{1 - \tilde{y}^2} \tilde{\Pi}_3 y(t), \quad \|\tilde{y}(t)\| = 1, \quad (2.19)$$

where

$$\begin{aligned} \tilde{\Pi}((y_1, y_2)) &= (\Pi_1, \Pi_2) \left(y_1, y_2, \sqrt{1 - y_1^2 - y_2^2} \right) \\ \tilde{\Pi}_3((y_1, y_2)) &= \Pi_3 \left(y_1, y_2, \sqrt{1 - y_1^2 - y_2^2} \right). \end{aligned}$$

Next define

$$\begin{aligned} f_1((y_1, y_2)) &= \Pi_1 \left(y_1, y_2, \sqrt{1 - y_1^2 - y_2^2} \right) - y_1 g(y_1, y_2) \\ f_2((y_1, y_2)) &= \Pi_2 \left(y_1, y_2, \sqrt{1 - y_1^2 - y_2^2} \right) - y_2 g(y_1, y_2) \quad \text{and} \\ g((y_1, y_2)) &= ((y_1, y_2), \tilde{\Pi}((y_1, y_2))) + \sqrt{1 - \tilde{y}^2} \tilde{\Pi}_3(\tilde{y}(t)). \end{aligned}$$

so that

$$y_1(t) \frac{dy_2}{dt}(t) y_2(t) \frac{dy_1}{dt}(t) = y_1(t) \Pi_2 \left(\tilde{y}(t), \sqrt{1 - \tilde{y}^2(t)} \right) - y_2(t) \Pi_1 \left(\tilde{y}(t), \sqrt{1 - \tilde{y}^2(t)} \right).$$

Now using the value of Π_1 and Π_2 , it becomes

$$\frac{d}{dt} \left[\ln \left(\frac{y_2(t)}{y_1(t)} \right) \right] = \frac{r}{y_1 y_2} \{ (y_1 + y_2)^2 (1 - p) - y_2^2 \} + (h - g) \frac{\sqrt{1 - y_1^2 - y_2^2}}{(k \sqrt{1 - y_1^2 - y_2^2}) + y_1 + y_2} \quad (2.20)$$

From above, we conclude that if $p = 1$, $h < g$ no periodic solution can exist, because $\frac{d}{dt} \left[\ln \left(\frac{y_2(t)}{y_1(t)} \right) \right]$ cannot change sign. So finally we may summarize to the following proposition.

Proposition 2.3.1(II). When fractions (p) of the newborns with genotype x equal to 1 and if the predation rate of prey with genotype y is higher than the predation rate on prey with genotype x then any trajectory can reach the origin from the interior following a fixed direction.

Remark: The biological implication is that when the newborns with genotype x equal to 1 then fractions of the newborns with genotype y would be vanish, then prey with genotype x also gradually decrease and go to extinct for their obvious reason. Moreover the predation rate of prey with genotype y is higher than the predation rate on prey with genotype x then predator population may also extinct and this may be a cause of total extinction of the ecosystem.

2.3.2 The behaviour of the system around predator free equilibrium $E_1(x_1, y_1, 0)$

Lemma 1. (i) The equilibrium point E_1 is locally asymptotically stable if and only if $h < h^{[*]}$, where $h^{[*]} = \frac{mrpb - earg + eagpr + mkba + mar - marp}{ebpr}$.

(ii) The system around the boundary equilibrium $E_1(x_1, y_1, 0)$ is globally asymptotically stable when

$$m > m^{[*]} = \max\{h(x_1 + e), g(y_1 + e)\}. \quad (2.21)$$

Proof: (i) If J_1 represents the Jacobian matrix of the system

(2.1) around the boundary equilibrium point $E_1(x_1, y_1, 0)$, then its eigenvalues are

$$\lambda_1 = \frac{r(-pb - a + ap)}{b}, \lambda_2 = \frac{r(-pb - a + ap)}{a} \quad (2.22)$$

and $\lambda_3 = -\frac{ehbrp - earg + eagpr + mkba + mrpb + mar - marp}{kba + rpb + ar - arp}$. Clearly the first two eigenvalues are negative and third will be negative when $h < h^{[*]}$. Therefore, the system is locally asymptotically stable iff $h < h^{[*]}$.

(ii) **Proof.** Let $R_x^3 = [(x, y, z) \in R_x^+, x > 0, y > 0, z \geq 0]$, consider the scalar function $L : R_x^3 \rightarrow R$ defined by

$$L(t) = \int (x - x_1) dx + \int (y - y_1) dy + \int dz \quad (2.23)$$

The derivative of (2.23) is

$$\begin{aligned} \frac{dL}{dt} = & -(x + y)\{a(x - x_1)^2 + b(y - y_1)^2\} + \frac{z\{x((x_1 + e)h - m) + y((y_1 + e)g - m)\}}{kz + x + y} \\ & - \frac{\{hx^2 + gy^2\} + mk\}z}{(kz + x + y)} \leq 0. \end{aligned}$$

which correspond to the assumption (2.21). Also $\frac{dL}{dt} = 0$, only for $(x, y, z) = (x_1, y_1, 0)$. Thus, L is a Lyapunov function and by Lasalle theorem [cf. Hale (1969)] global stability follows. This is illustrated in Fig. 1 for parameter values given in the figure.

2.3.3 The behaviour of the system around co-existence equilibrium $E_2(x_2, y_2, z_2)$

Lemma. (i) The equilibrium point E_2 is locally asymptotically stable if and only if the following conditions are satisfied

$$\frac{gy_2}{kz_2 + y_2} < h < g, \quad \frac{rp}{x_2} < a < \frac{rp}{x_2} + \frac{hz_2}{(kz_2 + x_2 + y_2)^2}, \quad (2.24)$$

$$\left(\frac{r(1-p)}{y_2} + \frac{gz_2}{(kz_2 + x_2 + y_2)^2}\right) < b \quad \text{and} \quad L_1 < M_1 < N_1. \quad (2.25)$$

where

$$\begin{aligned} L_1 & \equiv \frac{gy_2^2 z_2 - \{by_2^2 + r(1-p)x_2\}(kz_2 + x_2 + y_2)^2}{ey_2 z_2 \{gkz_2 + x_2(g-h)\}}, \quad M_1 \equiv \frac{\{r(1-p) - by_2\}(kz_2 + x_2 + y_2)^2 + gy_2 z_2}{ez_2 \{hkz_2 + y_2(h-g)\}} \\ N_1 & \equiv \frac{gy_2(x_2 + y_2)}{-(ax_2^2 + rpy_2)(kz_2 + x_2 + y_2)^2 + hx_2^2 z_2} \end{aligned} \quad (2.26)$$

(ii) The system is permanent when $h < h^{[*]}$ $k > 1$ and conditions (2.24), (2.25) along with the conditions (2.5) after recalling the uniform boundedness of the system (2.1) .

(iii) The coexistence equilibrium E_2 is globally asymptotically stable if, recalling the boundedness property of the population and the uniformly persistency of the system,

$$\begin{aligned} & e g y_2 < h(e k z_2 + (1 + e) y_2 + x_2), e h x_2 < g(e k z_2 + (1 + e) x_2 + y_2), 3a + \left\{ \frac{r p y_2}{W x_2} + \frac{h}{k(k W_1 + W)} \right\} \\ & > b + \left\{ \frac{h + g}{k^2 c} - \frac{r}{W} \right\}, (k + 2) h c (k z_2 + x_2 + y_2) \left\{ (a + b) x_2 e + (k e z_2 + (1 + e) y_2 + x_2) (b \right. \\ & \left. + \frac{r(1 - p) x_2}{W y_2}) \right\} + h z_2 \left\{ e x_2 (h + g) + g (k e z_2 + (1 + e) y_2 + x_2) \right\} \\ & < (k W_1 + W) (k z_2 + x_2 + y_2) \left\{ g(a + b) (k e z_2 + (1 + e) x_2 + y_2) \right. \\ & \left. + \frac{e x_2 h r}{c} + e b g y_2 + \frac{r(1 - p) x_2 g e}{c_2} \right\} + g z_2 \left\{ (h + g) (k e z_2 + (1 + e) x_2 + y_2) + y_2 g \right\}, \end{aligned} \quad (2.27)$$

$$\begin{aligned} & \left\{ a + b + \frac{z_2 (h + g)}{(k W_1 + W) (k z_2 + x_2 + y_2)} \right\} \left\{ \frac{(e g y_2 - h(e k z_2 + (1 + e) y_2 + x_2))}{(k W_1 + W) (k z_2 + x_2 + y_2)} \right\} \\ & < \left\{ \frac{r(e g y_2 - h(e k z_2 + (1 + e) y_2 + x_2))}{c^2 (k + 2) (k z_2 + x_2 + y_2)} \right\} + 2 \left[\left\{ a + \frac{r p y_2}{c x_2} + \frac{h z_2}{c(k + 2) (k z_2 + x_2 + y_2)} \right\} \right. \\ & \left. \left\{ \frac{h e x_2 - g(k e z_2 + (1 + e) x_2 + y_2)}{c(k + 2) (k z_2 + x_2 + y_2)} \right\} \right]. \end{aligned} \quad (2.28)$$

Proof: (i) It is noted that, J_2 as the Jacobian matrix evaluated at the coexistence equilibrium point $E_2(x_2, y_2, z_2)$, $\det(J_2)$ be the determinant of J_2 , $\text{tr}(J_2)$ be the trace of J_2 . The sum of second order principal minors of J_2 is denoted by M_2 . The Jacobian matrix at E_2 is given by $J_2 = (\phi_{ij})$, $i, j = 1, 2, 3$ with entries are shown in Appendix A.3.

For the conditions (2.24), the sign of $\phi_{12}, \phi_{31}, \phi_{32}$ are positive and of ϕ_{11} is negative. on the other hand for the conditions (2.25) the entries ϕ_{21}, ϕ_{22} has the sign negative. Also $\phi_{13}, \phi_{23}, \phi_{33}$ has the negative sign without any conditions. It follows that

$$\begin{aligned} \text{tr}(J_2) &= \phi_{11} + \phi_{22} + \phi_{33}, \\ \det(J_2) &= \phi_{13}(\phi_{32}\phi_{21} - \phi_{31}\phi_{22}) + \phi_{23}(\phi_{31}\phi_{12} - \phi_{32}\phi_{11}) + \phi_{33}(\phi_{11}\phi_{22} + \phi_{21}\phi_{12}), \\ M_2 &\equiv (\phi_{33} + \phi_{22})\phi_{11} + \phi_{22}\phi_{33} - \phi_{13}\phi_{31} - \phi_{12}\phi_{21} - \phi_{23}\phi_{32}, \\ C_2 &\equiv \text{tr}(J_2)M_2 - \det(J_2) \\ &= \phi_{11}^2\phi_{22} + \phi_{22}^2\phi_{11} - \phi_{11}\phi_{13}\phi_{31} - \phi_{22}\phi_{12}\phi_{21} - \phi_{22}\phi_{32}\phi_{23} - \phi_{21}\phi_{13}\phi_{32} - \phi_{12}(\phi_{11}\phi_{21} + \phi_{31}\phi_{23}) \\ &\quad + \phi_{33}(\phi_{11}\phi_{33} + \phi_{22}\phi_{33} + \phi_{11}^2 + 2\phi_{11}\phi_{22} + \phi_{22}^2 - \phi_{13}\phi_{31} - \phi_{23}\phi_{32}). \end{aligned}$$

Now clearly $M_2 > 0$, $\text{tr}(J_2) < 0$ and $\det(J_2) < 0$ if $(\phi_{32}\phi_{21} - \phi_{31}\phi_{22}) > 0$. But this follows from the second conditions of (2.25) Finally, we see that $C_2 < 0$ if $\left(\phi_{11}\phi_{21} + \phi_{31}\phi_{23} \right) > 0$ which also follows from the second conditions of (2.25). Hence the Routh-Hurwitz conditions hold so that all the characteristic roots of J_2 have negative real parts. E_2 is thus locally asymptotically stable.

Proof (ii): It has been shown that the system (2.1) is repeller around E_0 using the conditions $k > 1$ and 2nd part of first condition of (2.24). The boundary equilibrium E_1 is unstable when $h < h^{[*]}$ and the sufficient conditions of local asymptotic stability of the system are (2.24), (2.25). Moreover the system is dissipative when (2.5) is satisfied after recalling the uniform boundedness

of the system. Now from the main theorem of Butler et al. (1986) the system is uniformly persistent. And hence the system is permanent since it is uniformly persistent and uniformly bounded.

Proof (iii). Since the system is uniformly persistence, so there exists $c > 0$ such that $x(t), y(t), z(t) \geq c$. Now we consider the scalar function $L : R_x^3 \rightarrow R$

$$L(t) = \left(x - x_2 - x_2 \ln \frac{x}{x_2} \right) + \left(y - y_2 - y_2 \ln \frac{y}{y_2} \right) + \left(z - z_2 - z_2 \ln \frac{z}{z_2} \right) \quad (2.29)$$

with derivative along solution trajectories of (2.1) which are given by

$$\begin{aligned} \frac{dL_1}{dt} &= -(x - x_2)^2 \left(a + \frac{rpy_2}{xx_2} + \frac{hz_2}{(kz + x + y)(kz_2 + x_2 + y_2)} \right) + (x - x_2)(y - y_2) \left(\frac{rp}{x} - a \right. \\ &\quad \left. - \frac{hz_2}{(kz + x + y)(kz_2 + x_2 + y_2)} \right) + (x - x_2)(z - z_2) \frac{h(x_2 + y_2)}{(kz + x + y)(kz_2 + x_2 + y_2)}, \\ \frac{dL_2}{dt} &= -(y - y_2)^2 \left(b + \frac{r(1-p)x_2}{yy_2} + \frac{z_2g}{(kz_2 + x_2 + y_2)(kz + x + y)} \right) + (x - x_2)(y - y_2) \left(\frac{r(1-p)}{y} \right. \\ &\quad \left. - b - \frac{z_2g}{(kz_2 + x_2 + y_2)(kz + x + y)} \right) + (y - y_2)(z - z_2) \frac{(x_2 + y_2)g}{(kz_2 + x_2 + y_2)(kz + x + y)}, \\ \frac{dL_3}{dt} &= -e(z - z_2)^2 \frac{(h k x_2 + g y_2 k)}{(kz_2 + x_2 + y_2)(kz + x + y)} + e(x - x_2)(z - z_2) \frac{(h k z_2 + h y_2 - g y_2)}{(kz_2 + x_2 + y_2)(kz + x + y)} \\ &\quad + e(y - y_2)(z - z_2) \frac{(k g z_2 - h x_2 + g x_2)}{(kz_2 + x_2 + y_2)(kz + x + y)}, \end{aligned}$$

and which can be written in matrix notation as

$$\frac{dL}{dt} = -U^T M U.$$

where

$$M = \begin{bmatrix} A & H & G \\ H & B & F \\ G & F & C \end{bmatrix}. \quad (2.30)$$

The entries now are

$$\begin{aligned} A &= a + \frac{rpy_2}{xx_2} + \frac{hz_2}{(kz + x + y)(kz_2 + x_2 + y_2)}, \quad B = b + \frac{r(1-p)x_2}{yy_2} + \frac{z_2g}{(kz_2 + x_2 + y_2)(kz + x + y)}, \\ C &= e \frac{(h k x_2 + g y_2 k)}{(kz_2 + x_2 + y_2)(kz + x + y)}, \quad H = -\frac{1}{2} \left(\frac{rp}{x} - a - \frac{(h + g)z_2}{(kz + x + y)(kz_2 + x_2 + y_2)} + \frac{r(1-p)}{y} - b \right), \\ F &= \frac{1}{2} \left(\frac{e\{h x_2 - g(x_2 + k z_2)\} - (x_2 + y_2)g}{(kz_2 + x_2 + y_2)(kz + x + y)} \right), \quad G = \frac{1}{2} \left(\frac{e\{g y_2 - h(y_2 + k z_2)\} - (x_2 + y_2)h}{(kz_2 + x_2 + y_2)(kz + x + y)} \right) \end{aligned}$$

Its principal minors can be shown to be all positive. In fact

$P_1 = A(x) > 0$ for $x \geq 0$ obviously.

Now by 3rd condition of (2.27)

$$\begin{aligned} P_2 &= AB - H^2 \\ &= \left\{ \left(a + \frac{rpy_2}{xx_2} + \frac{hz_2}{(kz + x + y)(kz_2 + x_2 + y_2)} \right) \left(b + \frac{r(1-p)x_2}{yy_2} + \frac{z_2g}{(kz_2 + x_2 + y_2)(kz + x + y)} \right) \right\} \\ &\quad - \frac{1}{4} \left(\frac{rp}{x} - a - \frac{(h + g)z_2}{(kz + x + y)(kz_2 + x_2 + y_2)} + \frac{r(1-p)}{y} - b \right)^2 > 0. \end{aligned}$$

Next $G < 0$, $F < 0$ by 1st and 2nd conditions of (2.27). For $P_3 = (AB - H^2)C + F(GH - AF) + G(FH - BG)$, we have by (2.28),

$$\begin{aligned} GH - AF &= \frac{1}{4} \left(\left(\frac{e\{gy_2 - h(y_2 + kz_2)\} - (x_2 + y_2)h}{(kz_2 + x_2 + y_2)(kz + x + y)} \right) \left(\frac{rp}{x} - a - \frac{(h + g)z_2}{(kz + x + y)(kz_2 + x_2 + y_2)} \right) \right. \\ &\quad \left. + \frac{r(1-p)}{y} - b \right) \\ &\quad - \frac{1}{2} \left(a + \frac{rpy_2}{xx_2} + \frac{hz_2}{(kz + x + y)(kz_2 + x_2 + y_2)} \right) \left(\frac{e\{hx_2 - g(x_2 + kz_2)\} - (x_2 + y_2)g}{(kz_2 + x_2 + y_2)(kz + x + y)} \right) \\ &< 0 \end{aligned}$$

and $F(GH - AF) > 0$. Further for P_3 , we have by the fourth of (2.27)

$$\begin{aligned} &= -\frac{1}{4} \left[\left(\frac{e\{hx_2 - g(x_2 + kz_2)\} - (x_2 + y_2)g}{(kz_2 + x_2 + y_2)(kz + x + y)} \right) \left(\frac{rp}{x} - a - \frac{(h + g)z_2}{(kz + x + y)(kz_2 + x_2 + y_2)} + \frac{r(1-p)}{y} - b \right) \right] \\ &\quad - \frac{1}{2} \left[\left(b + \frac{r(1-p)x_2}{yy_2} + \frac{z_2g}{(kz_2 + x_2 + y_2)(kz + x + y)} \right) \left(\frac{e\{gy_2 - h(y_2 + kz_2)\} - (x_2 + y_2)h}{(kz_2 + x_2 + y_2)(kz + x + y)} \right) \right] < 0 \end{aligned}$$

and using $G < 0$ we find $G(FH - BG) > 0$, so that combining these results $P_3 > 0$. Hence M is positive definite, the derivative of $L(t)$ is negative along the trajectories and vanishes at E_2 . Thus, $L(t)$ is a Lyapunov function and Hence by Lyapunov-Lasalle's invariance principle [cf. Hale (1969)], E_2 is globally asymptotically stable. The Fig. 2 shows this fact for parameter values given in the figure.

2.4 Non-existence periodic solutions around $E_2(x_2, y_2, z_2)$

In this section, we would like to prove that under some suitable conditions, there is no periodic solution of the system (2.1) around the positive interior equilibrium $E_2(x_2, y_2, z_2)$. To prove this, the following technique by Li and Muldowney (1993), Arino et al. (2004) and Rahman and Chakravarty (2013b) can be applied. Consider the general autonomous ordinary differential equation

$$\frac{dX}{dt} = F(X(t)) \quad (2.31)$$

where F is a C^1 function in some open subset of R^N . Denoting by $J = (\frac{\partial F}{\partial X})$ the Jacobian matrix of (2.31) and $J^{[2]}$ the $\begin{pmatrix} N \\ 2 \end{pmatrix} \times \begin{pmatrix} N \\ 2 \end{pmatrix}$ matrix which is the second additive compound matrix associated the Jacobian matrix J (see appendix(A.4) for more details)

If $X \in R^N$ then the corresponding logarithmic norm of $J^{[2]}$ (that we denote by $\mu_\infty(J^{[2]})$) endowed by the vector norm $|X|_\infty = \sup_i |X_i|$ is

$$\mu_\infty(J^{[2]}) = \sup \left\{ \frac{\partial F_r}{\partial x_r} + \frac{\partial F_s}{\partial x_s} + \sum_{q \neq r, s} \left(\left| \frac{\partial F_q}{\partial x_r} \right| + \left| \frac{\partial F_q}{\partial x_s} \right| \right) : 1 \leq r < s \leq n \right\} \quad (2.32)$$

where $\mu_\infty(J^{[2]}) < 0$ implies the diagonal dominance by row matrix $J^{[2]}$. Then, the following result holds.

Theorem. A simple closed rectifiable curve that is invariant with respect to system (2.1) cannot exist if $\mu_\infty(J^{[2]}) < 0$.

The logarithmic norm μ_∞ , endowed by the vector norm $|X|_\infty$ of the second additive compound matrix $J^{[2]}$ associated with the Jacobian matrix J of the system (2.1), calculated on E_2 , is negative iff, recalling the uniformly persistency of the system the suprema of the following functions satisfy

$$(r - 3(a + b)c) + \frac{z\{e(h - g)(y - x) - gx - hy - z\{(1 - e)k(h + g)\}\}}{(kz + x + y)^2} < 0, \quad (2.33)$$

$$(r - (3a + b)c) + \frac{x^2(eh - m) + y^2(g(e + 1) - m) + xy(e(g + h) - 2m)}{(kz + x + y)^2} + \frac{z\{y(g(ek + 1) - h - 2km) + xk(eh - 2m) - zk(h + mk)\}}{(kz + x + y)^2} < 0, \quad (2.34)$$

$$(r - (a + 3b)c) + \frac{x^2(e + 1)h + (x^2 + y^2)(eg - m) + xy\{(e + 1)h\} - 2m + eg}{(kz + x + y)^2} + \frac{z\{x((e(1 - k) + 1)h - g - 2km) - 2mky\} - zk(g + mk)}{(kz + x + y)^2} < 0. \quad (2.35)$$

Sufficient conditions to satisfy (2.33), (2.34) and (2.35) are (i) $r < (3a + b)c$, $a < b$, (ii) $g > h(1 + e)$ (iii) $m > g(e + 1)$,

A direct application of Li and Muldowney's criteria shows that under the conditions (i), (ii) and (iii) there is no periodic solution for the present system (2.1), under consideration.

3 The predator free subsystem

By putting $z = 0$ in the system (2.1), the predator free system is as follows

$$\frac{dx}{dt} = (rp - ax)(x + y), \quad (3.1a)$$

$$\frac{dy}{dt} = (r(1 - p) - by)(x + y), \quad (3.1b)$$

$$x(0) > 0, \quad y(0) > 0. \quad (3.1c)$$

The equilibrium at the $\widehat{E}_0(0, 0)$ is an unstable steady state of the system (3.1) having eigenvalues 0 and r . The boundary equilibrium point $\widehat{E}_1(x_1, y_1)$, where x_1, y_1 are same values with earlier, is a stable node as the eigenvalues of the corresponding Jacobian matrix are same as λ_1 and λ_2 of equation (2.22).

3.1 Global stability for the predator free subsystem

Proof. Let $R_x^2 = [(x, y) \in R_x^+, x > 0, y > 0]$, consider the scalar function $L : R_x^2 \rightarrow R$ defined by

$$W = \left(x - x_2 - x_2 \ln \frac{x}{x_2}\right) + \left(y - y_2 - y_2 \ln \frac{y}{y_2}\right) \quad (3.2)$$

The derivative of (3.2) along the solutions of (3.1) is

$$\frac{dW}{dt} = \frac{dW_1(x, y)}{dt} + \frac{dW_2(x, y)}{dt}$$

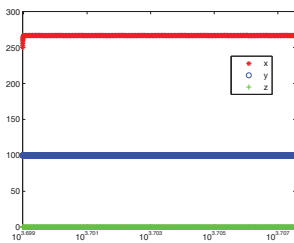
The derivative of (3.2) is

$$\frac{dW}{dt} = -(x + y)\{a(x - x_1)^2 + b(y - y_1)^2\} \leq 0.$$

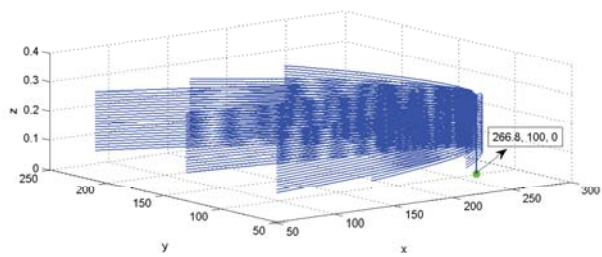
unconditionally. Also $\frac{dL}{dt} = 0$, only for $(x, y) = (x_1, y_1)$. Thus, W is a Lyapunov function and by Lasalle theorem [cf. Hale (1969)] the system is globally asymptotically stable unconditionally.

4 Simulations:

In this section we have established numerical simulation for the feasible equilibrium of the updated system (2.1) under consideration. I summarize all the conditions of local stability, global stability for the various equilibria in Table 2 and the default values of the parameters are shown in Table 1 for the convenience of the reader. The system is globally asymptotically stable at $E_1(266.8, 100, 0)$ when $h=0.00002$ and $g=0.00003$ with the default values in Table 1 satisfying the theoretical condition $m^{[*]} = \max\{0.00536, 0.00303\} = 0.00536 < 0.19 = m$ are shown in the figure 1 having eigen values $(-0.1898, -7.3333, -11)$. Next one increase gradually the value of crucial parameter up to 0.4 then bifurcation diagram have plotted with the successive maxima and minima of the prey and predator population with h as a variable parameter in figure 4(b-d). The quantitative behavior of the equilibrium point changes as the density of the prey population decline and of the predator increases. After variation of predation rate h at 2 of genotype x along with $g = 3$, the predation rate with genotype y the system reach Globally asymptotically stable point $E_2(262.2, 96.31, 387)$ having eigenvalues $(-0.1743, -7.3613, -10.7965)$ validating the theoretical conditions $egy_2 = 285.46284 < 8554.44856 = h(ekz_2 + (1 + e)y_2 + x_2)$ and $ehx_2 = 518.1072 < 13323.3708 = g(ekz_2 + (1 + e)x_2 + y_2)$ in figure 2



(a) The time series solution of Global stability at E_1 .



(b) The Phase portrait solution of Global stability at E_1 .

Figure 1: (a) Figure of global stability of the system (2.1) at the predator free equilibrium $E_1(266.8, 100, 0)$ (in green circle) with $r = 10, p = 0.8, k = 10, h = 0.00002, a = 0.03, b = 0.02, g = 0.00003, e = 0.988, m = 0.19$. We obtain that $m^{[*]} = \max\{0.00536, 0.00303\} = 0.00536 < 0.19 = m$.

Table 1: The table describes the definitions of the system parameters, their dimension and default values chosen for the simulations

Variable/ parameter	Units	Defination	Default value
x	number per unit designated area	prey density of genotypes x	
y	number per unit designated area	prey density of genotypes y	
z	number per unit designated area	density of predator	
r	per day	reproduction rate of prey	10
a	per day	population pressure felt by genotypes x	0.03
b	per day	population pressure felt by genotypes y	0.02
p	per day	fractions of the newborns	0.8
k		prey saturation constant	10
e		conversion factor	0.988
h	per day	hunting rate of predator on prey of genotypes x	
g	per day	hunting rate of predator on prey of genotypes y	
m	per day	mortality rate of predator	0.19

Table 2: Schematic representation : LAS/GAS=(Locally/Globally) asymptotically stable.

Equilibria	Conditions For Stability	Equilibrium nature	Figure
E_0	Details in Sec 2.3.1	conditionally stable	
E_1	$h < h^{[*]}$	LAS	
E_1	$m > m^{[*]} = \max\{h(x_1 + e), g(y_1 + e)\}$	GAS	Fig.1
E_2	conditions (2.24) and (2.25)	LAS	
E_2	conditions (2.27) and (2.27)	GAS	Fig.2
$\hat{E}_0$	unstable	unconditionally unstable	
$\hat{E}_1$	unconditionally stable	LAS $\Rightarrow$ GAS	

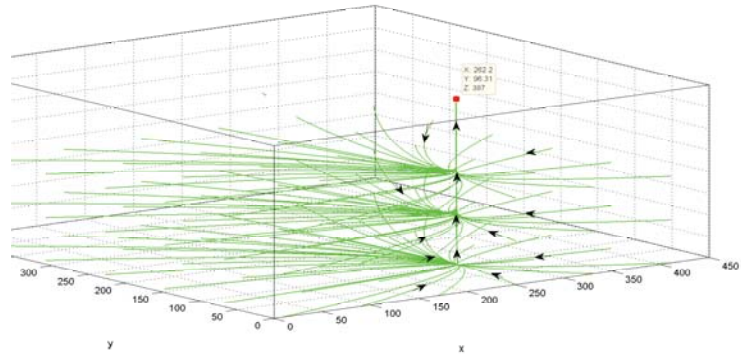
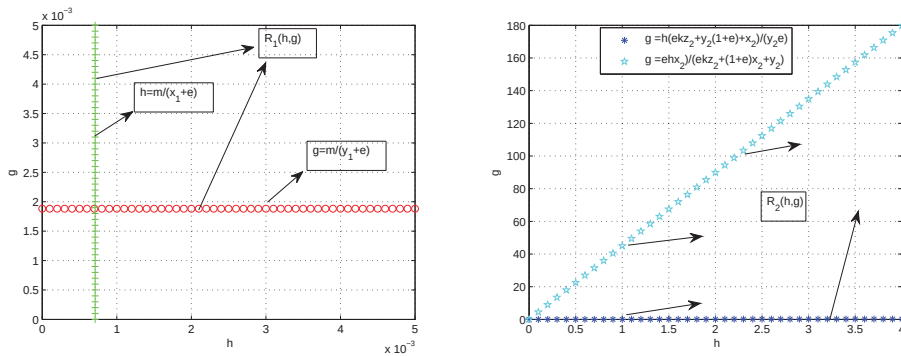


Figure 2: The phase portrait of the global stability of the system (2.1) around $E_2(262.2, 96.31, 387)$ (in red circle) with parameter values $r = 10, p = 0.8, k = 10, h = 2, a = 0.03, b = 0.02, g = 3, e = 0.988, m = 0.19$. It is verified that $egy_2 = 285.46284 < 8554.44856 = h(ekz_2 + (1 + e)y_2 + x_2)$ and $ehx_2 = 518.1072 < 13323.3708 = g(ekz_2 + (1 + e)x_2 + y_2)$



(a) Parametric sub regions for the global stability of the equilibrium E_1 . (b) Parametric sub regions for the global stability of the equilibrium E_2 .

Figure 3: (a) Figure of parametric restrictions having sub region $R_1 = \{(h, g) : m > m^{[*]} = \max\{h(x_1 + e), g(y_1 + e)\}\}$ and $R_2 = \{(h, g) : egy_2 < h(ekz_2 + (1 + e)y_2 + x_2), ehx_2 < g(ekz_2 + (1 + e)x_2 + y_2)\}$ of global stability of system (2.1) around E_1 and E_2 respectively in the h - g parameter space with The values of the other parameters are shown in Table2.

5 Discussion

The current article describe an eco-genetic system formed by the three non autonomous Ordinary Differential Equations. The first two represent two genetically distinguished prey population and the third one represents their common predators in their common ecological environment. The eco-genetic model (2.1) proposed here is an innovative model of its kind, since the genetically differentiated prey population is consumed by the predator population with ratio dependent response function which is a improved path and updated notion in population Biology. The updated model is a modification of the model proposed by Rahman (2016).

The behaviour of the system around the feasible equilibria has been analyzed. The suitable

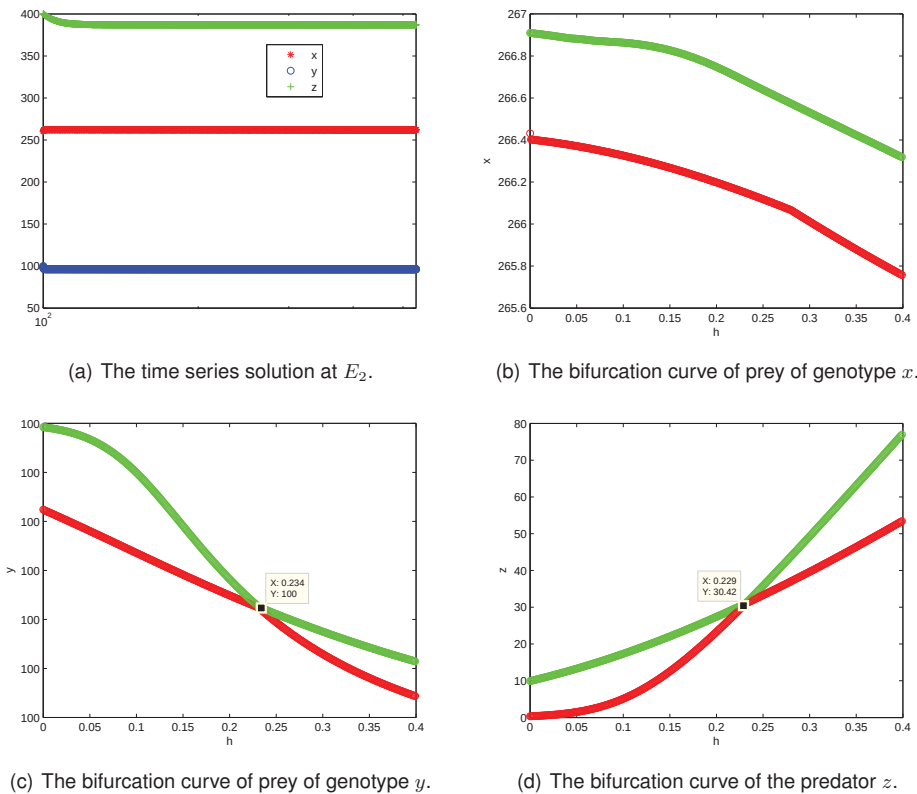


Figure 4: (a) Figure of global stability of system (2.1) around E_2 with same data values of figure 2. (b)-(d) bifurcation diagram of the system (2.1) for increasing value of h and other parameter values are the same as of figure 1.

restrictions of the crucial parameters as for example h, g, p have been performed for which the updated system cannot reach the origin following a fixed direction or spirally. The most interesting dynamics are shown near the positive coexistence equilibrium $E_2(x_2, y_2, z_2)$. It is described the local stability, global stability and bifurcation situations of the proposed model (2.1). Constructing two Lyapunov functions defined in the equations (2.23) and (2.29), I have shown the global stability of the system (2.1) around E_1 and E_2 under suitable parametric conditions respectively in the parametric space h - g (see Figure 3). The Bannixson's criterion in R^n have been applied to deduce the conditions of non-existence of periodic solutions around coexistence equilibrium E_2 . Another remarkable feature noticed here that it is not possible to have equilibrium with just one genotype. This outcome means that new genotypes can arise from an original genotype which is clear in the reproduction terms of the system compare the first two equations of (2.1). As a result both x and y population save offsprings also belonging to the other population. it will be eventually restock by the mutations occurring in the other one.

Next the influence of predator population to the proposed model (2.1) have been shown by constructing predator free sub system (3.1) in section 3.1. This subsystem (3.1) has two steady states, $\hat{E}_0(0, 0)$ and $\hat{E}_1(x_1, y_1)$. At the predator free subsystem $\hat{E}_0$ has its eigenvalues $r, 0$. This shows that the predator free subsystem is unstable. This characteristic remains unchanged also on the introduction of the predators in whole system studied by Rahman (2016) and Viberti and

Venturino (2014) that means the ecosystem cannot disappear which differ the dynamics of the updated model (2.1). Total extinction is possible here with the suitable parametric restriction. Also $\hat{E}_1$ is globally asymptotically stable unconditionally which have been proved by applying suitable Lyapunov function (3.2). This proof have been done in the previous article Rahman (2016) by using Dulac criterion.

Thus we may conclude that the predator population in a eco-genetic ratio dependent predator-prey system may act as a crucial role for destabilizing the equilibrium between the two genetically different strains of prey which is the same situation in the study of Viberti and Venturino (2014) with genetic diversity of its predator population. The biological implication of the updated system is that when the newborns with genotype x equal to 1 then fractions of the newborns with genotype y would be vanish, then prey with genotype x also gradually decrease and go to extinct for their obvious reason. Moreover the predation rate of prey with genotype y is higher than the predation rate on prey with genotype x then predator population may also extinct and this may be a cause of total extinction of the ecosystem. Hence the ecosystem is affected by genetically distinguished population in a alternative way. This depends on which trophic level population lies.

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A Appendix

A.1 The values of y_2 and $\theta_1, \theta_2, \theta_3, \theta_4$

$$\begin{aligned}
 y_2 = & -hr \left(-gm^2hrp - r^2keghmp + h^2mx_2kerp^2b - r^2p^2ke^2hg^2 + he^2g^2r^2kp + 2heg^2mrp \right. \\
 & - h^2mx_2kebrp - ah^2mx_2^2kebp - ah^2mx_2^2kebp + a^2ghmx_2^2kep + r^2p^2ke^2g^3 - he^2g^3rp - h^2m^2x_2b \\
 & + g^3mrp^2e - g^2hmerp^2 - x_2hm^2bpg + 2gh^2mx_2eb - aghm^2x_2p - 2ag^2hm^2x_2e - ax_2g^3pem \\
 & - g^2mker^2p^2 + ghmker^2p^2 - g^2m^2rp^2 + ghm^2rp^2 + h^2m^2x_2bp - g^2x_2h^2e^2b + ag^3x_2he^2 \\
 & + aghm^2x_2 + ax_2g^2m^2p + a^2g^3x_2ke^2 + g^2e^2x_2hkbrp - ghmx_2kerp^2b - ag^2x_2hke^2r + x_2hmbpeg^2 \\
 & + ah^2mx_2keb - gh^2mx_2epb - a^2g^2mx_2^2kep - a^2ghmx_2^2ke - 2ag^3x_2ke^2rp - ag^2x_2^2ke^2hb \\
 & + ag^2hm^2x_2ep + ag^2x_2ke^2hrp + aghmx_2ker + ag^2mx_2kerp + ag^2mx_2kerp^2 - aghmx_2kerp^2 \\
 & \left. + aghmx_2^2kebp \right) / \left(-g^3hke^2br^2p^2 + g^2h^2ke^2br^2p^2 + h^2m^2brpg - 2g^2erph^2bm - ag^2hm^2rp \right. \\
 & - g^2e^2h^2kbr^2p + 2ag^3herpm + ag^3ke^2r^2ph - ag^2hmker^2p + g^2x_2h^3e^2b^2 + a^3x_2^2ke^2g^4 + a^2g^4x_2e^2h \\
 & + a^2g^2x_2hm^2 + x_2h^3b^2m^2 + g^3h^2e^2brp - ag^4he^2rp + ag^4r^2p^2ke^2 - 2a^2g^3x_2ehm - 2gx_2h^3eb^2m \\
 & - 2agx_2h^2bm^2 - 2ag^3x_2h^2e^2b - g^2x_2h^2ke^2b^2rp - a^3x_2^2kehmg^2 + gh^2kebr^2mp - ag^3e^2r^2p^2kh \\
 & + x_2k^3kebr^2rpm + ag^2x_2^2h^2ke^2b^2 + 4ag^2x_2h^2ebm + 3ag^3x_2hke^2brp - 2a^2g^4x_2ke^2rp - a^2g^3x_2ke^2hr \\
 & + a^2g^3x_2ke^2rph - 2a^2g^3x_2hke^2b + a^2g^2x_2kehrm + a^2g^2x_2krephm + 2a^2gx_2h^2kebm \\
 & \left. - ax_2h^3ke^2bm + ag^2x_2h^2ke^2br - ag^2x_2h^2ke^2brp - 2agx_2h^2kebrpm - agx_2h^2kebm \right)
 \end{aligned}$$

and

$$\begin{aligned}
 \theta_1 &= g^2kea^3 + h^2keb^2a - 2ghkeba^2, \\
 \theta_2 &= -h^2keb^2rp - g^2kea^2r - 2g^2kerpa^2 + ghkerpa^2 - g^2ma^2 - h^2mb^2 - ghkea^2r \\
 &\quad - 2gh^2eba + h^2kebra + ghkebra + g^2hea^2 + 2ghmba + eb^2h^3 + 3ghkebrpa - h^2kebrpa, \\
 \theta_3 &= gh^2ebr + ghmrpa - ghker^2p^2a - ghker^2p^2b - h^2kebr^2p + h^2mbrp - ghkebr^2p + 2g^2mrpa, \\
 &\quad + g^2ker^2p^2a + gh^2erpb - 2ghmbrp - h^2mbr + ghker^2a - ghmrpa - g^2herpa \quad (A.1) \\
 &\quad + h^2ker^2p^2b - g^2hear + 2g^2ker^2pa, \\
 \theta_4 &= -g^2mr^2p^2 - ghker^3p + g^2her^2p - g^2ker^3p^2 + ghmr^2p^2 - ghmr^2p + ghker^3p^2
 \end{aligned}$$

A.2 The Jacobian matrix

The Jacobian matrix $J \equiv DF(s)$ of the system (2.1) at any arbitrary point (x, y, z) is given by

$$J(x, y, z) = (\delta_{ij})_{3 \times 3}$$

where

$$\begin{aligned} \delta_{11} &= -a(x+y) + rp - ax - \frac{hz}{kz+x+y} + \frac{hxx}{(kz+x+y)^2}, \\ \delta_{12} &= rp - ax + \frac{hxx}{(kz+x+y)^2}, \quad \delta_{13} = -\frac{hx}{kz+x+y} + \frac{hxxk}{(kz+x+y)^2}, \\ \delta_{22} &= -b(x+y) + r(1-p) - by - \frac{gz}{kz+x+y} + \frac{gyz}{(kz+x+y)^2}, \\ \delta_{21} &= r(1-p) - by + \frac{gyz}{(kz+x+y)^2}, \quad \delta_{31} = \frac{ezh}{kz+x+y} - \frac{ez(hx+gy)}{(kz+x+y)^2}, \\ \delta_{23} &= -\frac{gy}{kz+x+y} + \frac{gyzk}{(kz+x+y)^2}, \quad \delta_{32} = \frac{ezg}{kz+x+y} - \frac{ez(hx+gy)}{(kz+x+y)^2}, \\ \delta_{33} &= \frac{e(hx+gy)}{kz+x+y} - \frac{ez(hx+gy)k}{(kz+x+y)^2} - m. \end{aligned}$$

A.3 All entries of J_2

The Jacobian at the pt E_2 of the system (2.1) is $J_2 = (\phi_{ij})$, $i, j = 1, 2, 3$ with entries

$$\begin{aligned} \phi_{11} &= -a(x_2+y_2) + rp - ax_2 - \frac{hz_2}{kz_2+x_2+y_2} + \frac{hx_2z_2}{(kz_2+x_2+y_2)^2}, \\ \phi_{12} &= rp - ax_2 + \frac{hx_2z_2}{(kz_2+x_2+y_2)^2}, \quad \phi_{13} = -\frac{hx_2}{kz_2+x_2+y_2} + \frac{hx_2z_2k}{(kz_2+x_2+y_2)^2}, \\ \phi_{22} &= -b(x_2+y_2) + r(1-p) - by_2 - \frac{gz_2}{kz_2+x_2+y_2} + \frac{gy_2z_2}{(kz_2+x_2+y_2)^2}, \\ \phi_{21} &= r(1-p) - by_2 + \frac{gy_2z_2}{(kz_2+x_2+y_2)^2}, \quad \phi_{31} = \frac{ez_2h}{kz_2+x_2+y_2} - \frac{ez_2(hx_2+gy_2)}{(kz_2+x_2+y_2)^2}, \\ \phi_{23} &= -\frac{gy_2}{kz_2+x_2+y_2} + \frac{gy_2z_2k}{(kz_2+x_2+y_2)^2}, \quad \phi_{32} = \frac{ez_2g}{kz_2+x_2+y_2} - \frac{ez_2(hx_2+gy_2)}{(kz_2+x_2+y_2)^2}, \\ \phi_{33} &= -\frac{ez_2(hx_2+gy_2)k}{(kz_2+x_2+y_2)^2}. \end{aligned}$$

A.4 Theoretical Conditions for non existence of periodic solution

The definition of the second additive compound matrix can be established in the paper of Li and Muldowney (1993). Let $J = (\xi_{ij})$ be an $n \times n$ matrix. The second additive compound matrix $A^{[2]}$ is the $\binom{N}{2} \times \binom{N}{2}$ matrix defined as follows, for any integer $i = 1, 2, \dots, \binom{N}{2}$, let $(i) = (i_1, i_2)$ be the i th number in the lexicographic ordering of integer pairs (i_1, i_2) such that

$1 \leq i_1 < i_2 \leq n$. Then, the element in the i th row and j th column of $J^{[2]}$ is
 $= \xi_{i_1 i_1} + \xi_{i_2 i_2}$, if $(j) = (i)$
 $= (-1)^{r+s} \xi_{i_r j_s}$, if exactly one entry i_r of (i) does not occur in (j) and j_s does not occur in (i)
 $= 0$, if neither entry from (i) occurs in (j) .
 For $n = 3$, $J = [\xi_{ij}]$ $i, j = 1, 2, 3$ the second additive computed matrix $J^{[2]}$ is given by

$$J^{[2]} = \begin{bmatrix} \xi_{11} + \xi_{22} & \xi_{23} & -\xi_{13} \\ \xi_{32} & \xi_{11} + \xi_{33} & \xi_{12} \\ -\xi_{31} & \xi_{21} & \xi_{22} + \xi_{33} \end{bmatrix}.$$

In this case, $(1) = (1, 2)$, $(2) = (1, 3)$ and $(3) = (2, 3)$.

Theorem (Bendixson's criterion in R^n). A simple closed rectifiable curve that is invariant with respect to system (2.31) cannot exist if any one of the following conditions is satisfied on R^n :

$$(i) \sup \left\{ \frac{\partial F_r}{\partial x_r} + \frac{\partial F_s}{\partial x_s} + \sum_{q \neq r, s} \left(\left| \frac{\partial F_q}{\partial x_r} \right| + \left| \frac{\partial F_q}{\partial x_s} \right| \right) : 1 \leq r < s \leq n \right\} < 0, \quad (\text{A.2})$$

$$(ii) \sup \left\{ \frac{\partial F_r}{\partial x_r} + \frac{\partial F_s}{\partial x_s} + \sum_{q \neq r, s} \left(\left| \frac{\partial F_r}{\partial x_q} \right| + \left| \frac{\partial F_s}{\partial x_q} \right| \right) : 1 \leq r < s \leq n \right\} < 0, \quad (\text{A.3})$$

$$(iii) \lambda_1 + \lambda_2 < 0, \quad (\text{A.4})$$

$$(iv) \inf \left\{ \frac{\partial F_r}{\partial x_r} + \frac{\partial F_s}{\partial x_s} - \sum_{q \neq r, s} \left(\left| \frac{\partial F_q}{\partial x_r} \right| + \left| \frac{\partial F_q}{\partial x_s} \right| \right) : 1 \leq r < s \leq n \right\} > 0, \quad (\text{A.5})$$

$$(v) \inf \left\{ \frac{\partial F_r}{\partial x_r} + \frac{\partial F_s}{\partial x_s} - \sum_{q \neq r, s} \left(\left| \frac{\partial F_r}{\partial x_q} \right| + \left| \frac{\partial F_s}{\partial x_q} \right| \right) : 1 \leq r < s \leq n \right\} > 0, \quad (\text{A.6})$$

$$(vi) \lambda_{n-1} + \lambda_n > 0, \quad (\text{A.7})$$

where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ are the eigenvalues of $\frac{1}{2}((\frac{\partial F}{\partial x})^* + \frac{\partial F}{\partial x})$ and where $\frac{\partial F}{\partial x}$ is the Jacobian matrix of F while the asterisk denotes transposition.