

Dynamics of SIR Model with Saturated Treatment Function under the Effect of Pollution

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ABSTRACT

This paper studies SIR model with incidence rate which is saturated and nonlinear. We have considered saturated treatment function on a logistically growing population. Boundedness of the system has been obtained along with existence of disease-free and endemic equilibrium points. We deduce that the strict threshold for illness obliteration is weak when the basic reproduction number approaches unity. The global stability of the endemical equilibrium point is established using Lyapunov Lasalle theorem. Condition for Persistence of endemicity of our model has been discussed. Mathematically this paper recommends that keeping the atmosphere clean, building healthy relation with nature, giving timely treatment to the patients by increasing the equipment and capacity of hospitals, improving the cure efficiency are all valid methods for managing the disease. Numerical simulations are shown that exhibit results obtained analytically.

Keywords: Equilibrium Point, Saturated Treatment Function, Non Linear Incidence Rate, Bifurcation, Stability Analysis, Persistence.

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1 Introduction

Mathematical modeling in epidemiology have come to be effective and crucial device to apprehend the infectious disease dynamics and to enhance control of infection in the population. An accurate epidemic model is an intelligent model which is capable of detecting any possible outbreak of the disorder and is powerful in decreasing the transmission of the disease. It is an easy version of real world biology (Brauer and Castillo-Chavez, 2001), ((Anderson and May, 1991), (Britton, 2003)).

In literature on population dynamics, ecology and epidemiology, numerous mathematical models have been suggested ((Takeuchi, Ma and Beretta, 2000), (Wang, 2006), (Diekmann and Heesterbeek, 2000), (Chauhan, Bhatia and Gupta, 2015), (Arino, McCluskey and Driessche, 2003), (Haderler and Driessche, 1997), (Driessche and Watmough, 2000), (Diaz and Bhatta, 2018), (Reddy, Narayan and Reddy, 2018), (Abta, Alaoui and Kaddar, 2011)). Nowadays, the major concerning issue is communicable diseases among human beings which is spreading rapidly

like flu, cold, amoebiasis, malaria, influenza, and athlete's foot. Several scholars researched the epidemic models that demonstrate the complex behavior of communicable disease transmission ((Cherry, Reeves and Smith, 1998),(Diekmann and Heesterbeek, 2000),(Thomas, Miller and Gwanyama, 2018)).

Nowadays, due to increase in toxicants and pollutants at an alarming rate the ecosystem gets imbalanced. Pollutants can be carbon emissions from industry, burning the e-waste under the high temperature, mining, deforestation at a large scale, use of fossil fuels for vehicular transportation and farming activities. As a result, mathematical modeling of ecotoxicology was key research area in early 1980s. Due to continuous increase in environmental pollution, biological parameters in the ecosystem like recruitment rate, natural diminishing rate, recovery rate and disease associated diminishing rate has a certain degree of fluctuation ((Xiao and Ruan, 2007),(Hethcote, 1995),(Wendi and Zhien, 2002),(Chauhan, Bhatia and Sharma, 2018)).

The fundamental research topics for the epidemic model systems are the presence of threshold value that is reproduction number, which asserts whether or not the communicable disease will perish ((Thieme, 1992),(Cherry et al., 1998)). Nevertheless, the backward bifurcation trend has drawn engrossment in the field of disease control in recent years. Since all countries and urban areas have upper limit for permissible range for the cure of illness, Wang ((Wang, 2006)) presented a staged cure work which delineates that the treatment rate corresponds to the infective count until the treatment rate has reached the saturation point.

Further the description of our paper is: Section 2 formulates the mathematical model followed by discussion on the boundedness of our model in section 3. Existence and stability of equilibrium points has been studied in section 4. The global stability of equilibrium points of our model has been discussed in the subsequent section. Persistence of endemicity of our model has been analyzed in section 6. Section 7 has numerical examples to validate our theoretical findings. Finally we conclude the results for our model in last section.

2 Mathematical model

We have relied on the following assumptions in our mathematical model:

- (1) We study logistically growing population with carrying capacity or recruitment rate $H \in R^+$. We assume that infected and recovered do not assist the births and deaths in susceptible. We observe that the susceptible gets infected when in contact with infected individuals with γ as the infection rate. Also, the susceptible population $S(t)$ becomes infected due to pollution factor ' U^* ' and ' α_1 ' is a constant.

$$\frac{dS}{dt} = S(H - S) - \gamma SI - \alpha_1 US \quad (2.1)$$

- (2) The infected class $I(t)$ is formed when the susceptible population comes in contact with the infected. There is a decline in the rate of infected class due to ' d ' the natural diminishing rate, ' μ ' the natural recovery rate and ' σ ' the disease related diminishing rate of individuals:

$$\frac{dI}{dt} = \gamma SI - (d + \sigma + \mu + \alpha_2 U)I \quad (2.2)$$

- (3) The infected class will also come in contact with the pollution and with the timely treatment, infected class can recover with help of saturated treatment function $\left(\frac{rI}{1+\alpha I}\right)$, where ' r ' is known as the healing rate and $\left(\frac{1}{1+\alpha I}\right)$ depicts the opposite impact of infected being deferred for treatment, here, ' r ' is positive and ' α ' is non-negative. The modified equation is:

$$\frac{dI}{dt} = \gamma SI - (d + \sigma + \mu + \alpha_2 U)I - \frac{rI}{1 + \alpha I} \quad (2.3)$$

- (4) Similarly, the recovered class $R(t)$ can be formulated as:

$$\frac{dR}{dt} = \mu I + \frac{rI}{1 + \alpha I} - dR - \alpha_3 UR \quad (2.4)$$

where ' α_3 ' is non-negative.

- (5) The concentration of environmental pollutant $X(t)$ is the difference between pollutant's constant exogenous input rate Q into the surroundings and loss rate of toxicant from environment hX .

$$\frac{dX}{dt} = Q - hX \quad (2.5)$$

- (6) The organismal concentration of toxicant $U(t)$ is

$$\frac{dU}{dt} = a_1 X + \frac{d_1 \eta \phi}{a_1} - (l_1 + l_2)U \quad (2.6)$$

where a_1 is the intake rate per unit mass organism of environmental pollutant, η is the pollutant concentration in the resource, α_2 is the intake rate per unit mass of organism of the food, d_1 is the intake rate per unit mass organism of toxins in food, l_1 and l_2 are net ingestion of organism and pollutant's deputation rate respectively.

Thus, the SIR model formulated is as follows:

$$\begin{cases} \frac{dS}{dt} = S(H - S) - \gamma SI - \alpha_1 US \\ \frac{dI}{dt} = \gamma SI - (d + \sigma + \mu + \alpha_2 U)I - \frac{rI}{1 + \alpha I} \\ \frac{dR}{dt} = \mu I + \frac{rI}{1 + \alpha I} - dR - \alpha_3 UR \\ \frac{dX}{dt} = Q - hX \\ \frac{dU}{dt} = a_1 X + \frac{d_1 \eta \phi}{a_1} - (l_1 + l_2)U \end{cases} \quad (2.7)$$

Consider a system of differential function of three variables as follows:

$$\frac{da}{dt} = p(a, b) \quad (2.8)$$

$$\frac{db}{dt} = q(b) \quad (2.9)$$

where both p and q are locally Lipschitz and continuous in a in R^n and solutions for all positive time exists. Equation (2.8) act as the limit equation for the equation (2.9) which is asymptotically autonomous if $p(t, a) \rightarrow q(a)$ as $t \rightarrow \infty$ uniformly for all a in R^n . Using the corollary of Thieme (Thieme, 1992) our reduced model is provided below:

$$\begin{cases} \frac{dS}{dt} = S(H - S) - \gamma SI - \alpha_1 U^* S \\ \frac{dI}{dt} = \gamma SI - (d + \sigma + \mu + \alpha_2 U^*) I - \frac{rI}{1 + \alpha I} \\ \frac{dR}{dt} = \mu I + \frac{rI}{1 + \alpha I} - dR - \alpha_3 U^* R \end{cases} \quad (2.10)$$

As initial two equations do not depend on R , therefore we study the following model:

$$\begin{cases} \frac{dS}{dt} = S(H - S) - \gamma SI - \alpha_1 U^* S \\ \frac{dI}{dt} = \gamma SI - (d + \sigma + \mu + \alpha_2 U^*) I - \frac{rI}{1 + \alpha I} \end{cases} \quad (2.11)$$

3 Boundedness

The boundedness of our model has been proved in the provided lemma.

Lemma 3.1. For the region:

$$B = \left\{ (S, I, R) \in R_+^3, 0 \leq S \leq H, 0 \leq S + I + R \leq \frac{H^2}{k} \right\}$$

entire solutions will lie in this region as $t \rightarrow \infty$ and $k = \max(\alpha_1 U^*, d, \alpha_3 U^*, \sigma + \alpha_2 U^*)$.

Proof. : Let:

$$V(t) = S(t) + I(t) + R(t)$$

From (2.10), we get,

$$\dot{V}(t) \leq H^2 - \alpha_1 S U^* - d(I + R) - (\sigma + \alpha_2 U^*) I - \alpha_3 U^* R$$

Consider, $k = \max(\alpha_1 U^*, d, \alpha_3 U^*, \sigma + \alpha_2 U^*)$ then $\dot{V}(t) \leq H^2 - k(S + I + R)$

Then, by using comparison theorem we have,

$$V(t) \leq \frac{H^2}{k}$$

□

4 Equilibrium Points and Stability Analysis

4.1 Disease Free Equilibrium

Here, we shall obtain equilibrium points of our system and a parameter called reproduction number (R_0). In the current model, R_0 is as given below:

$$R_0 = \frac{\gamma(H - \alpha_1 U^*)}{d + \sigma + \mu + \alpha_2 U^* + r}$$

The disease free equilibrium point of system (2.11) is expressed as, $E_0 = (S_0, I_0) = (H - \alpha_1 U^*, 0)$. The stability analysis of E_0 of our model is studied as follows:

$$J(E_0) = \begin{pmatrix} -(H - \alpha_1 U^*) & -\gamma(H - \alpha_1 U^*) \\ 0 & \gamma(H - \alpha_1 U^*) - (d + \sigma + \mu + \alpha_2 U^* + r) \end{pmatrix}$$

One of the eigenvalues of $J(E_0)$ is $-(H - \alpha_1 U^*)$ which is negative and other is negative if $R_0 < 1$. Thus, susceptible individuals will not catch infection if and only if $R_0 < 1$. Thus, the theorem:

Theorem 4.1. E_0 is locally asymptotically stable if $R_0 < 1$ and is unstable if $R_0 > 1$.

4.2 Endemic Equilibrium

Here, we shall obtain the endemic equilibrium point. The endemic equilibria of our model (2.11) are solutions of,

$$\begin{cases} H - S - \gamma I - \alpha_1 U^* = 0 \\ \gamma S - (d + \sigma + \mu + \alpha_2 U^*) - \frac{r}{1 + \alpha I} = 0 \end{cases} \quad (4.1)$$

Use the elimination method to remove the variable S from the second equation of (4.1) by extracting its value from the first equation of (4.1) we get,

$$lI^2 + mI + n = 0 \quad (4.2)$$

where the variables l , m and n are as follows:

$$\begin{aligned} l &= \gamma^2 \alpha \\ m &= \gamma(\gamma - \alpha H + \alpha \alpha_1 U^*) + \alpha(d + \sigma + \mu + \alpha_2 U^*) \\ n &= -\gamma(H - \alpha_1 U^*) + (d + \sigma + \mu + \alpha_2 U^* + r) \end{aligned}$$

Now, The following cases can be deduced from the equation (4.2).

Case I: If $\alpha = 0$, the equation (4.2) is linear and $I = -\frac{n}{m}$ is its unique solution if and only if $n < 0$ or R_0 is greater than 1. Therefore, unique endemic equilibrium prevails for $R_0 > 1$ which approaches zero as $R_0 \rightarrow 1+$. Also, existence of unique endemic equilibrium when R_0 less than 1 is not possible. Further, backward bifurcation at R_0 equals to 1 is not possible.

Case II: If $\alpha > 0$, the equation (4.2) is quadratic in nature and there exists unique endemic equilibrium for R_0 greater than 1. Also, when $n = 0$ or $R_0 = 1$ there exists unique endemic equilibrium where $I = -\frac{m}{l}$ which will be positive if and only if $m < 0$.

Two positive equilibria exists on an interval to the left of $R_0 = 1$ ($R^c < R_0 < 1$) because

equilibria depend continuously on R_0 , that is,

$$I_1 = \frac{-m - \sqrt{m^2 - 4ln}}{2l} \text{ and } I_2 = \frac{-m + \sqrt{m^2 - 4ln}}{2l}$$

Since (4.2) is a quadratic equation in I , we can solve the discriminant value $m^2 - 4ln = 0$ in terms of R_0 such that $R_0 = R^c$. Now, substituting the values in the discriminant we get,

$$R^c = \frac{\gamma(H^c - \alpha_1 U^*)}{(d + \sigma + \mu + \alpha_2 U^* + r)}$$

$$\text{where, } H^c = \frac{\gamma\alpha\alpha_1 U^* + \alpha(d + \sigma + \mu + \alpha_2 U^*) - \gamma^2}{\gamma\alpha} + \frac{2\sqrt{\alpha r}}{\alpha}$$

Thus, the theorem:-

Theorem 4.2. 1. For $\alpha = 0$, equation (4.2) is linear, has a unique solution that is, $I = -\frac{n}{m}$ and therefore, unique endemic equilibrium exists if $R_0 > 1$ and endemic equilibrium doesn't exist for $R_0 < 1$.

2. For $\alpha > 0$

(A) The system (2.11) has an endemic equilibrium which is unique when $R_0 > 1$.

(B) At $n = 0$ or $R_0 = 1$, equation (4.2) has a unique positive root $I = -\frac{m}{l}$ which is positive if and only if $m < 0$.

(C) The system (2.11) has two endemic equilibria $E_1(S_1, I_1)$ and $E_2(S_2, I_2)$ under the condition $R^c < R_0 < 1$ and $m < 0$, where $I_1 = \frac{-m - \sqrt{m^2 - 4ln}}{2l}$ and $I_2 = \frac{-m + \sqrt{m^2 - 4ln}}{2l}$

(D) The system (2.11) has no endemic equilibria whenever $R_0 < R^c$ and $m < 0$ or whenever $R_0 < 1$ and $m > 0$.

Theorem 4.3. System (2.11) has backward bifurcation when $R_0 = 1$ if and only if $m < 0$.

Proof. Let $y = h(x) = lx^2 + mx + n$. When $n = 0$ implying that $R_0 = 1$, the curve passes through the $(0, 0)$. Further, in case $m < 0$, we have $h(x) = 0$ has root greater than 0. Next, we increase n to strictly greater than 0, considering $h(x)$ is continuous function of n as a fact that ensures which there exists an open interval of n , $(0, \sigma)$ say, on which $h(x) = 0$ has two real roots which are positive. Thus, existence of two endemic equilibria are possible when $R_0 < 1$. Also requirement is apparent that for the system (2.11), there does not exist positive real roots if $m \geq 0$, in this manner completing the confirmation. $\square$

Next we can express the parameters $d, \sigma, \mu, \alpha_2 U^*, r, \alpha_1$ in terms of α to show its explicit criteria as well as backward bifurcation at $n = 0$ that is $R_0 = 1$ with the help of the equation,

$$\gamma(H - \alpha_1 U^*) = (d + \sigma + \mu + \alpha_2 U^* + r) \quad (4.3)$$

The condition $m < 0$ is,

$$\begin{aligned} \gamma(\gamma - \alpha H + \alpha\alpha_1 U^*) + \alpha(d + \sigma + \mu + \alpha_2 U^*) &< 0 \\ \gamma^2 + \gamma\alpha\alpha_1 U^* + \alpha(d + \sigma + \mu + \alpha_2 U^*) &< \gamma\alpha H \end{aligned}$$

Substitute the value of γH from (4.3)

$$\gamma^2 + \gamma\alpha\alpha_1 U^* + \alpha(d + \sigma + \mu + \alpha_2 U^*) < \alpha(d + \sigma + \mu + \alpha_2 U^* + r + \gamma\alpha_1 U^*)$$

The equation reduces to,

$$\alpha > \frac{\gamma^2}{r} \triangleq \alpha_0 \quad (4.4)$$

We observe that at $R_0 = 1$ there is a backward bifurcation when γH is provided by (4.3) if and only if (4.4) is holds true. It can be noted that backward bifurcation will occur when the effect of delay in treatment of infective is more than some level. For the given system (2.11), α is one of the parameters leading to the backward bifurcation.

There will be no positive solutions of (4.2) in the case that $n > 0$ and either $m \geq 0$ or $m^2 < 4ln$, and thus there are no endemic equilibria. There is a possibility of existence of two positive solutions, comparing to endemic equilibria E_1 and E_2 of (4.2), if and only if $n > 0$ (or $R_0 < 1$), and $m < 0$, $m^2 > 4ln$. If $m = -2\sqrt{ln}$ there exists one solution of (4.2) where $I = -\frac{m}{2l}$ and solution is positive.

When $\alpha > \alpha_0$ holds true, there will be two endemic equilibria where the values lie in a form of interval ranging from R^c which is characterized by $m = -2\sqrt{ln}$ to $R_0 = 1$. We substitute the values of l, m and n in the formula $m^2 = 4ln$ to induce a quadratic equation in H in order to find the value of R^c .

$$\gamma^2 \alpha^2 H^2 - 2\gamma\alpha(-\gamma^2 + \gamma\alpha\alpha_1 U^* + \alpha(d + \sigma + \mu + \alpha_2 U^*))H + ((-\gamma^2 + \gamma\alpha\alpha_1 U^* + (\alpha(d + \sigma + \mu + \alpha_2 U^*)))^2 - 4\gamma^2 \alpha r) = 0$$

$$\text{Therefore, } H = \frac{\gamma\alpha\alpha_1 U^* + \alpha(d + \sigma + \mu + \alpha_2 U^*) - \gamma^2}{\gamma\alpha} \pm \frac{2\sqrt{\alpha r}}{\alpha}$$

Since, $m < 0$ and $R_0 < 1$ here so we get,

$$\frac{\alpha(d + \sigma + \mu + \alpha_2 U^* + r) + \gamma^2 + \gamma\alpha\alpha_1 U^*}{\gamma\alpha} H < \frac{(d + \sigma + \mu + \alpha_2 U^* + r) + \gamma\alpha_1 U^*}{\gamma}$$

Thus, we can get,

$$H^c = \frac{\gamma\alpha\alpha_1 U^* + \alpha(d + \sigma + \mu + \alpha_2 U^*) - \gamma^2}{\gamma\alpha} + \frac{2\sqrt{\alpha r}}{\alpha}$$

So,

$$R^c = \frac{\gamma(H^c - \alpha_1 U^*)}{(d + \sigma + \mu + \alpha_2 U^* + r)}$$

Now, the following theorem shall be proved:

Theorem 4.4. *The local asymptotic stability of unique endemic equilibrium is achieved if $\alpha < \min \{\alpha_0, \gamma_0\}$ when $R_0 > 1$.*

Proof. Now, we shall obtain the conditions under which endemic equilibrium is stable which exists if $R_0 > 1$. Consider,

$$J = \begin{pmatrix} H - 2S - \gamma I - \alpha_1 U^* & -\gamma S \\ \gamma I & \gamma S - (d + \sigma + \mu + \alpha_2 U^*) - \frac{r}{(1 + \alpha I)^2} \end{pmatrix}$$

From equation (4.1) we have,

$$\gamma S = (d + \sigma + \mu + \alpha_2 U^*) + \frac{r}{(1 + \alpha I)} \quad (4.5)$$

Using (4.5) we get, $\gamma S - (d + \sigma + \mu + \alpha_2 U^*) - \frac{r}{(1 + \alpha I)^2} = \frac{r\alpha I}{(1 + \alpha I)^2}$

$$\text{So, } J = \begin{pmatrix} H - 2S - \gamma I - \alpha_1 U^* & -(d + \sigma + \mu + \alpha_2 U^*) - \frac{r}{(1 + \alpha I)} \\ \gamma I & \frac{r\alpha I}{(1 + \alpha I)^2} \end{pmatrix}$$

The determinant of the above matrix is obtained as follows:-

$$\det J = -\frac{Sr\alpha I}{(1 + \alpha I)^2} + \gamma^2 SI$$

This will be positive if and only if $\gamma^2 SI > \frac{Sr\alpha I}{(1 + \alpha I)^2}$ which is possible if $\alpha < \frac{\gamma^2}{r}$

So, $\det J > 0$ if

$$\alpha < \frac{\gamma^2}{r} = \alpha_0 \quad (4.6)$$

We have seen that (2.11) has no backward bifurcation under this condition. Now consider,

$$\text{tr } J = -S + \frac{r\alpha I}{(1 + \alpha I)^2} = -\left(\frac{d + \sigma + \mu + \alpha_2 U^*}{\gamma}\right) - \frac{r}{\gamma(1 + \alpha I)} + \frac{r\alpha I}{(1 + \alpha I)^2}$$

which is negative if

$$\left(\frac{d + \sigma + \mu + \alpha_2 U^*}{\gamma}\right) + \frac{r}{\gamma(1 + \alpha I)} > \frac{r\alpha I}{(1 + \alpha I)^2}$$

which is possible if, $\alpha < \frac{1}{\gamma} \triangleq \gamma_0$

□

We shall now obtain conditions under which two endemic equilibrium points are locally stable.

Theorem 4.5. 1. E_2 is locally asymptotically stable when $2(d + \sigma + \mu + \alpha_2 U^*) > \gamma r$.

2. E_1 is saddle when,

$$6\gamma^4 + 2\alpha^r(d + \sigma + \mu + \alpha_2 U^* + r) + 2\sqrt{\Delta}m + 4\alpha r < 2\alpha^2(d + \sigma + \mu + \alpha_2 U^*)^2 + 4\alpha\gamma^2(d + \sigma + \mu + \alpha_2 U^*) + 4\gamma^2 + 4\sqrt{\Delta} + 8\gamma^2 r\alpha.$$

Proof. Two endemic equilibria are E_1, E_2 exists when $R_0 < 1$, and let J_i be the Jacobian matrix at $E_i, i = 1, 2$, then we get

$$\det J_2 = \left[H - 2\left(\frac{d + \sigma + \mu + \alpha_2 U^*}{\gamma}\right) - \frac{2r}{\gamma(1 + \alpha I_2)} - \gamma I_2 - \alpha_1 U^* \right] \frac{r\alpha I_2}{(1 + \alpha I_2)^2} + \gamma I_2(d + \sigma + \mu + \alpha_2 U^*) + \frac{\gamma r I_2}{(1 + \alpha I_2)}$$

$$\text{i.e. } \det J_2 = \frac{-Sr\alpha I_2}{(1 + \alpha I_2)^2} + \gamma I_2(d + \sigma + \mu + \alpha_2 U^*) + \frac{\gamma I_2 r}{(1 + \alpha I_2)}$$

which is positive if and only if

$$\frac{-Sr\alpha I_2}{(1 + \alpha I_2)^2} + \gamma I_2(d + \sigma + \mu + \alpha_2 U^*) + \frac{\gamma I_2 r}{(1 + \alpha I_2)} > 0$$

Consider,

$$Sr\alpha I_2 + \gamma I_2(d + \sigma + \mu + \alpha_2 U^*)(1 + \alpha I_2)^2 + \gamma I_2 r(1 + \alpha I) > -Sr\alpha + \gamma(d + \sigma + \mu + \alpha_2 U^*) + \gamma r$$

$\therefore \det J_2 > 0$ if $S\alpha - \gamma > 0$ which is possible if $\alpha(d + \sigma + \mu + \alpha_2 U^*) > \gamma^2$

Note that $I_2 = \frac{-m + \sqrt{m^2 - 4ln}}{2l} > -\frac{m}{2l}$ and $m^2 - 4ln \geq 0$. Using these two conditions, one can easily check that, $\alpha(d + \sigma + \mu + \alpha_2 U^*) > \gamma^2$ and thus, $\det J_2 > 0$.

Now, $\text{tr } J_2 = -S + \frac{r\alpha I}{(1 + \alpha I)^2}$ which is negative if $S(1 + \alpha I)^2 > r\alpha I$

After substituting the value of S , it is verified that, $\text{tr } J_2 < 0$ if $2(d + \sigma + \mu + \alpha_2 U^*) > \gamma r$

Next, we show that the equilibrium E_1 is a saddle. Also, here, $I_1 = \frac{-m - \sqrt{m^2 - 4ln}}{2l}$. Consider,

$$\det J_1 = \frac{-Sr\alpha I}{(1 + \alpha I)^2} + \gamma^2 SI$$

$$\det J_1 \text{ will be negative if } \frac{Sr\alpha I}{(1 + \alpha I)^2} > \gamma^2 SI$$

$$\text{that is, if, } r\alpha > \gamma^2(1 + \alpha I)^2$$

Now, substituting the value of I_1 in above expression we get,

$$4\gamma^4 + 2b^2 + 2\sqrt{\Delta}b < 4ac + 4b + 4\sqrt{\Delta} + 4\gamma^2 r\alpha \quad (4.7)$$

Now, L.H.S of the above inequality is,

$$4\gamma^4 + 2\gamma^4 + 2\gamma^2\alpha^2(H - \alpha_1 U^*)^2 - 4\gamma^3\alpha(H - \alpha_1 U^*) + 2\alpha^2(d + \sigma + \mu + \alpha_2 U^*)^2 + 4\gamma^2\alpha(d + \sigma + \mu + \alpha_2 U^*) - 4\gamma\alpha^2(d + \sigma + \mu + \alpha_2 U^*)(H - \alpha_1 U^*) + 2\sqrt{\Delta}m \quad (4.8)$$

And R.H.S of inequality is,

$$-4\gamma^3\alpha(H - \alpha_1 U^*) + 4\gamma^2\alpha(d + \sigma + \mu + \alpha_2 U^* + r) + 4\gamma^2 - 4\gamma\alpha(H - \alpha_1 U^*) + 4\alpha(d + \sigma + \mu + \alpha_2 U^*) + 4\sqrt{\Delta} + 4\gamma^2 r\alpha \quad (4.9)$$

Using the conditions $R_0 < 1$ and $b < 0$ we get,

$$\gamma^2 + \alpha(d + \sigma + \mu + \alpha_2 U^*) < \gamma\alpha(H - \alpha_1 U^*) < \alpha(d + \sigma + \mu + r + \alpha_2 U^*) \quad (4.10)$$

Now, substituting equations (4.8), (4.9) in (4.7) we get,

$$6\gamma^4 + 2\gamma^2\alpha^2(H - \alpha_1 U^*)^2 + 2\alpha^2(d + \sigma + \mu + \alpha_2 U^*)^2 - 4\gamma\alpha^2(d + \sigma + \mu + \alpha_2 U^*)(H - \alpha_1 U^*) + 2\sqrt{\Delta}b - 4\gamma^2\alpha r - 4\gamma^2 + 4\gamma\alpha(H - \alpha_1 U^*) - 4\alpha(d + \sigma + \mu + \alpha_2 U^*) - 4\sqrt{\Delta} - 4\gamma^2 r\alpha < 0$$

Using (4.10) in the above inequality we get, E_1 is saddle if,

$$6\gamma^4 + 2\alpha^2(d + \sigma + \mu + \alpha_2 U^* + r) + 2\sqrt{\Delta}m + 4\alpha r < 2\alpha^2(d + \sigma + \mu + \alpha_2 U^*)^2 + 4\alpha\gamma^2(d + \sigma + \mu + \alpha_2 U^*) + 4\gamma^2 + 4\sqrt{\Delta} + 8\gamma^2 r\alpha$$

Thus, the theorem is proved. $\square$

5 Global Behaviors

We shall utilize the Lyapunov Lasalle theorem that allows an adequate condition under which our system (2.11) achieves global asymptotic stability for disease free equilibrium. Characterize R^d as,

$$R^d = 1 - \frac{r}{d + \sigma + \mu + r + \alpha_2 U^*} + \frac{1}{\alpha}$$

Theorem 5.1. *The disease free equilibrium E_0 achieves global asymptotic stability if $R_0 \leq \min\{1, R^d\}$.*

Proof. Let, $\mathbb{V}(S, I) = S - S_0 - S_0 \ln \frac{S}{S_0} + I$, then,

$$\begin{aligned}
 \frac{dV}{dt} &= S(H - S) - \gamma SI - \alpha_1 U^* S - \frac{S_0}{S} [S(H - S) - \gamma SI - \alpha_1 U^* S] + \gamma SI \\
 &\quad - (d + \sigma + \mu + \alpha_2 U^*) I - \frac{rI}{1 + \alpha I} \\
 &= S(H - S) - \alpha_1 U^* S - S_0 H + S S_0 + S_0 \alpha_1 U^* \\
 &\quad - \left(-\gamma S_0 + (d + \sigma + \mu + \alpha_2 U^*) + \frac{r}{1 + \alpha I} \right) I \\
 &= S(H - S - \alpha_1 U^*) - S_0 (H - \alpha_1 U^* - S) \\
 &\quad - \left(-\gamma (H - \alpha_1 U^*) + (d + \sigma + \mu + \alpha_2 U^*) + \frac{r}{1 + \alpha I} \right) I \\
 &= -(S_0 - S)^2 - \left(-\gamma (H - \alpha_1 U^*) + (d + \sigma + \mu + \alpha_2 U^*) + \frac{r}{1 + \alpha I} \right) I \\
 &= -(S_0 - S)^2 - \frac{g(I)}{1 + \alpha I} I
 \end{aligned}$$

where $(S - S_0)^2 \geq 0$ because square of any number is greater than or equal to 0.

Also, $g(I) = a_1 I + b_1$

where,

$$a_1 = -\gamma \alpha (H - \alpha U^*) + \alpha (d + \sigma + \mu + \alpha_2 U^*)$$

$$b_1 = -\gamma (H - \alpha U^*) + (d + \sigma + \mu + \alpha_2 U^* + r)$$

We observe that $b_1 \geq 0$ is equivalent to $R_0 \leq 1$, and $a_1 \geq 0$ is equivalent to $R_0 \leq R^d$, since

$$\begin{aligned}
 R_0 \leq R^d &\Rightarrow \frac{\gamma S_0}{d + \sigma + \mu + \alpha_2 U^* + r} \leq 1 - \frac{r}{d + \sigma + \mu + \alpha_2 U^* + r} + \frac{1}{\alpha} \\
 &\Rightarrow \frac{\gamma S_0}{d + \sigma + \mu + \alpha_2 U^* + r} \leq \frac{\alpha (d + \sigma + \mu + \alpha_2 U^* + r) - r \alpha + (d + \sigma + \mu + \alpha_2 U^* + r)}{\alpha (d + \sigma + \mu + \alpha_2 U^* + r)} \\
 &\Rightarrow \frac{\gamma S_0}{d + \sigma + \mu + \alpha_2 U^* + r} \leq \frac{\alpha (d + \sigma + \mu + \alpha_2 U^* + r) - r \alpha}{\alpha (d + \sigma + \mu + \alpha_2 U^* + r)} \\
 &\Rightarrow \alpha \gamma S_0 \leq \alpha (d + \sigma + \mu + \alpha_2 U^*) \\
 &\Rightarrow a_1 \geq 0
 \end{aligned}$$

So $g(I) \geq 0$ when $I \geq 0$ and if $R_0 \leq \min\{1, R^d\}$. Thus, $\frac{dV}{dt} \leq 0$ when $R_0 \leq \min\{1, R^d\}$.

Thus, by the Lyapunov-Lasalle theorem all solutions directly approach to E_0 when $R_0 \leq \min\{1, R^d\}$. $\square$

Now, we will inspect the global behavior of endemic equilibrium point when $R_0 > 1$.

Theorem 5.2. No limit cycle exists for system (2.11) when $\alpha\gamma < 1$ and $\alpha < 1$.

Proof. Let us consider, the Dulac function as $D = \frac{1+I}{I}$ to vanish off the limit cycle.

$$\begin{aligned}
 P(S, I) &= S(H - S) - \gamma SI - \alpha_1 U^* S \\
 Q(S, I) &= \gamma SI - (d + \sigma + \mu + \alpha_2 U^*) I - \frac{rI}{1 + \alpha I}
 \end{aligned}$$

$$\begin{aligned}
 PD &= \frac{S(H - S)(1 + I)}{I} - \gamma S(1 + I) - \frac{\alpha_1 U^* S(1 + I)}{I} \\
 QD &= \gamma S(1 + I) - (d + \sigma + \mu + \alpha_2 U^*)(1 + I) - r \frac{(1 + I)}{1 + \alpha I} \\
 \frac{\partial}{\partial S}(PD) &= \frac{H(1 + I)}{I} - \frac{2S(1 + I)}{I} - \gamma(1 + I) - \frac{\alpha_1 U^*(\alpha + I)}{I} \\
 \frac{\partial}{\partial I}(QD) &= \gamma S - (d + \sigma + \mu + \alpha_2 U^*) - \frac{r}{(1 + \alpha I)} + \frac{r\alpha}{(1 + \alpha I)^2} + \frac{r\alpha I}{(1 + \alpha I)^2}
 \end{aligned}$$

Then, according to the condition,

$$\frac{\partial(PD)}{\partial S} + \frac{\partial(QD)}{\partial I} = \frac{H(1 + I)}{I} - \frac{2S(1 + I)}{I} - \gamma(1 + I) - \frac{\alpha_1 U^*(\alpha + I)}{I} + \gamma S - (d + \sigma + \mu + \alpha_2 U^*) -$$

$$\begin{aligned} & \frac{r}{(1+\alpha I)} + \frac{r\alpha}{(1+\alpha I)^2} + \frac{r\alpha I}{(1+\alpha I)^2} \\ &= -S \frac{(1+I)}{I} + \gamma S - (d + \sigma + \mu + \alpha_2 U^*) - \frac{r}{(1+\alpha I)} + \frac{r\alpha}{(1+\alpha I)^2} + \frac{r\alpha I}{(1+\alpha I)^2} \end{aligned}$$

This will be negative if, $\frac{-r(1+\alpha I) + r\alpha\gamma + r\alpha\gamma I}{\gamma(1+\alpha I)^2} < 0$

if, $r(-1 + \gamma\alpha) + r\alpha I(-1 + \alpha) < 0$

if $-1 + \alpha\gamma < 0$ and $-1 + \alpha < 0$

that is, $\alpha\gamma < 1$ and $\alpha < 1$

$\therefore$ The system (2.11) has no limit cycle if $\alpha\gamma < 1$ and $\alpha < 1$. □

Thus, we have the theorem:

Theorem 5.3. *When $R_0 > 1$, the unique endemic equilibrium point is globally asymptotically stable if $\alpha < \min\{a_0, \gamma_0\}$ and $\alpha < 1$.*

Thus, we can say that if the delay in treatment for infective is reduced then unique endemic equilibrium exists and is globally asymptotically stable. Focusing on new medical advances, increasing the count of equipment and capacity of hospitals are all effective ways of reducing the delay in treatment.

6 Persistence of Endemicity

For an infinite dimensional system there exists the persistence theorem (Hale and Watman, 1998) for the uniform persistence of our system (2.11). Consider, $\mathbb{Y}$ to be a complete metric space. Assume that, $\mathbb{Y}^0, \mathbb{Y}_0 \subset \mathbb{Y}, \mathbb{Y}^0 \cap \mathbb{Y}_0 = \emptyset$ and $\mathbb{Y}^0 \cup \mathbb{Y}_0 = \mathbb{Y}$. Assume $\mathbb{J}(t)$ is a P_0 semigroup on $\mathbb{Y}$ satisfying

$$\mathbb{J}(t) : \mathbb{Y}^0 \rightarrow \mathbb{Y}^0$$

$$\mathbb{J}(t) : \mathbb{Y}_0 \rightarrow \mathbb{Y}_0$$

Let $\mathbb{J}_\partial(t) = \mathbb{J}(t)|_{\mathbb{Y}_0}$, $\mathbb{J}_0(t) = \mathbb{J}(t)|_{\mathbb{Y}^0}$, and A_∂ be the global attractor for $J_\partial(t)$.

Lemma 6.1. Let $\mathbb{J}(t)$ satisfies (2.10) and the following conditions listed below:

- (i) There is a t_0 which is positive such that $\mathbb{J}(t)$ is compact for t strictly greater than t_0 .
- (ii) $\mathbb{J}(t)$ is point dissipative in $\mathbb{Y}$.
- (iii) $\bar{A}_\partial = \bigcup_{\mathbb{Y} \in A_\partial} W(\mathbb{Y})$ is isolated and has an acyclic covering $M = \{M_1, M_2, \dots, M_3\}$.

Then $\mathbb{J}(t)$ is uniformly persistent if $W^S(M_i) \cap \mathbb{Y}^0 = \emptyset (i = 1, 2, \dots, n)$, where the stable manifold of invariable set is denoted by $W^S(M_i)$ and ω -limit set is given by $W(\mathbb{Y})$.

Now by defining: $\mathbb{Y} = \{(S, I) : S \geq 0, I \geq 0\}$, $\mathbb{Y}^0 = \{(S, I) : S \geq 0, I > 0\}$, $\mathbb{Y}_0 = \{(S, I) : S \geq 0, I = 0\}$. Therefore, $\mathbb{Y}^0$ is open and dense in Y with $\mathbb{Y}^0 \cup \mathbb{Y}_0 = \mathbb{Y}$ and $\mathbb{Y}^0 \cap \mathbb{Y}_0 = \emptyset$. Now, $E_1(H - \alpha U^*, 0)$ is to be shown as a weak repeller for $\mathbb{Y}^0$.

Theorem 6.2. *In the first quadrant, the system (2.11) is uniformly persistent when $R_0 \geq 1$.*

Proof. Lemma 2 shows the boundedness of the system (2.11). We observe that $\mathbb{Y}^0$ and $\mathbb{Y}_0$ are positive invariant and conditions of Lemma 11 are satisfied. $W^s(\mathbb{Y}_0) \cap \mathbb{Y}^0 = \emptyset$ shall be proved. $E_1(H - \alpha_1 U^*, 0)$ being the constant solution in Y_0 , exists. If for the initial condition in $\mathbb{Y}_0$, there exists a solution such that we get $(S(t), I(t)) \rightarrow (H - \alpha_1 U^*, 0)$ when $t \rightarrow \infty$, which implies that we have isolated and acyclic covering that is, $E_1(H - \alpha_1 U^*, 0)$. The existence of $(S(t), I(t))$ which is a solution greater than 0 with positive initial condition of an omega limit point which is E_1 so that $(S(t), I(t)) \rightarrow (H - \alpha_1 U^*, 0)$ when $t \rightarrow \infty$ is implied from the condition $W^s(\mathbb{Y}_0) \cap \mathbb{Y}^0 \neq \emptyset$. Thus, there exists $\epsilon > 0$ which is arbitrarily small when $R_0 > 1$ such that,

$$\epsilon_1 = \gamma(H - \alpha_1 U^*) - (d + \sigma + \mu + r + \alpha_2 U^*) - \epsilon > 0$$

We have,

$$H - \alpha_1 U^* - \epsilon < S(t) < H - \alpha_1 U^* + \epsilon$$

when t is big enough,

Hence,

$$\begin{aligned} \frac{dI}{dt} &= \gamma S - (d + \sigma + \mu + r + \alpha_2 U^*) \\ &> \gamma(H - \alpha_1 U^* - \epsilon) - (d + \sigma + \mu + r + \alpha_2 U^*) \\ &= \epsilon_1 I > 0 \end{aligned}$$

Thus, $\lim_{t \rightarrow \infty} I(t) \rightarrow \infty$, a contradiction to $\lim_{t \rightarrow \infty} I(t) \rightarrow 0$. Thus, it leads to $W^s(\mathbb{Y}_0) \cap \mathbb{Y}^0 = \emptyset$. When $\gamma(H - \alpha_1 U^*) - (d + \sigma + \mu + r + \alpha_2 U^*) = 0$, which is equal to $R_0 = 1$, $E_1(H - \alpha_1 U^*, 0)$, a saddle node, there does not exist any solution with initial condition greater than 0 which has an omega limit point E_1 . Hence, the proof is completed. $\square$

7 Numerical Section

Here, we will verify the mathematical results stated analytically in our model, for hypothetical and biologically feasible set of parameters: $\gamma = 0.01, d = 0.1, \mu = 0.1, r = 2, \alpha_2 = 0.5, \alpha_1 = 0.1$

1. For hypothetical and biologically feasible set of parameters $H = 2, \sigma = 0.01, \alpha = 0.5, U^* = 0.2$, reproduction number, R_0 is 0.0088 which is less than 1 and the conditions of Theorem 3 and 8 holds true. E_0 is globally asymptotically stable can be inferred from (Figure 1).

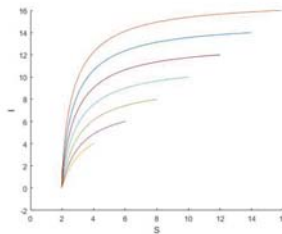


Figure 1: Graph of the system when $R_0 < 1$.

2. For set of parameters $H = 20, \gamma = 0.2, \sigma = 0.02, \alpha = 0.01, U^* = 0.2$ such that the $R_0 = 1.78$ which is greater than 1, unique endemic equilibrium is globally stable and Theorem 10 is verified (Figure 2).

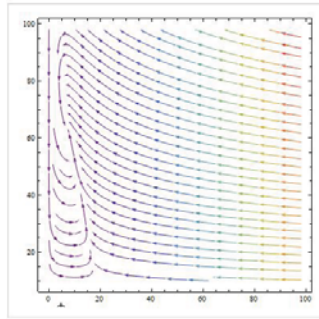


Figure 2: Graph of the system when $R_0 > 1$.

3. The graph shows infective sizes at equilibria versus R_0 for the set of parameters $\gamma = 0.1, \alpha_1 = 0.2, d = 1, \sigma = 0.1, \mu = 0.1, r = 2, \alpha = 0.5, \alpha_2 = 0.1, U^* = 0.2$, here α is large enough that leads to backward bifurcation with two endemic equilibria when $R_0 < 1$ which verifies Theorem 5 (Figure 3).

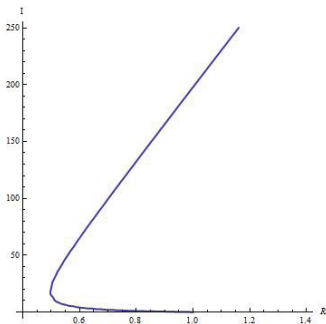


Figure 3: Graph of I versus R_0 .

4. The backward bifurcation is changing to forward when plotting infective sizes at equilibria versus R_0 for the set of parameters $\gamma = 0.5, \alpha_1 = 0.2, d = 1, \sigma = 0.1, \mu = 0.1, r = 2, \alpha = 0.1, \alpha_2 = 0.1, U^* = 0.2$ (Figure 4).

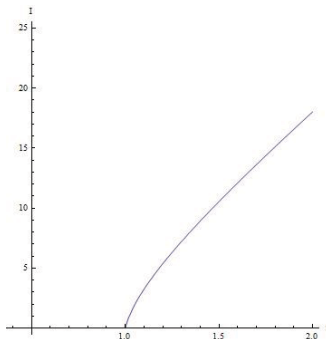


Figure 4: Graph of I versus R_0 .

5. When we increase the pollution to a large extent for the set of parameters $H = 50, \sigma = 0.1, r = 1, \alpha = 0.1, U^* = 10$ we observe that $S_0 = 49.006$ and infectives wipe out from the population gradually (Figure 5).

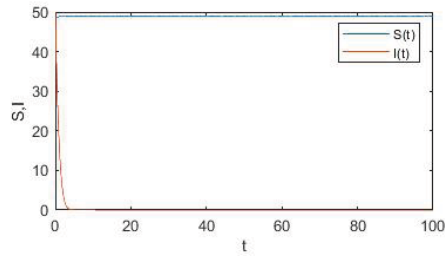


Figure 5: Graph of the system when $R_0 < 1$ and $U^* = 10$.

6. For $H = 50, \sigma = 0.1, \alpha = 0.1, U^* = 0.1$, we observe that susceptibles are less than the infectives. Next, we decrease the delay in the treatment parameter that is α to 0.05 and keeping all other parameters same we see that $S_0 = 49.9945$ and $I_0 = 0$ that is infectives are wiped out from the population. This is relevant biologically since giving timely treatment to patients by increasing the efficiency and decreasing the environmental pollution will decrease the infection. This can be inferred from Figure 6 and 7.

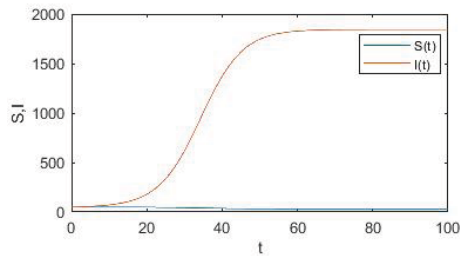


Figure 6: Graph of the system when $\alpha_1 = 0.1$ and $U^* = 0.1$.

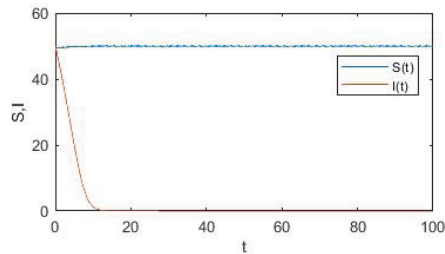


Figure 7: Graph of the system when $\alpha_1 = 0.05$ and $U^* = 0.1$.

Remark 7.1. Thus, we need to control the pollution in the environment as well as have technologies to cure the patients effectively.

8 Conclusion

Dynamics of an SIR model on logistically growing population under the effect of pollution has been studied. Global asymptotic stability of E_0 is achieved with the condition R_0 is less than 1. We have shown that, at $R_0 = 1$ there is backward bifurcation if the delay in treatment for

infective is more than certain threshold(α_0) if and only if $m < 0$. Thus, in order to wipe out the disease, bringing down R_0 to 1 is not sufficient. We need to invest more in medical advances in order to decrease the delay in the treatment. The system (2.11) has no limit cycle under the conditions $\alpha\gamma < 1$ and $\alpha < 1$ so, our system has a unique endemic equilibrium when R_0 is greater than 1, establishing its global asymptotic stability when $\alpha < \min\left\{\frac{\gamma^2}{r}, \frac{1}{\gamma}\right\}$ and extent of delayed treatment should be less than 1. From the numerical section, we explored that when pollution is sufficiently high, infectives wipe out from the population. But when pollution and delay in treatment parameter are decreased, susceptibles are increased and infectives are decreased thereby retaining the overall population. Thus, we need to decrease the pollution as well as increase the treatment capacity by investing on new medical advances and technologies to eradicate the disease.

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