

Dynamics of SIRS Epidemic Models under Saturated Treatment

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ABSTRACT

In this work we consider an SIRS epidemic model with a nonlinear incidence rate. By analyzing the model, the local stability of an endemic equilibrium point is discussed. The system is globally asymptotically stable if $R_0 > 1$. The analytical results are supported with numerical simulation using MATLAB and it is shown that the system is globally asymptotically stable.

Key words: Disease free, endemic equilibrium point, local stability, global stability

Mathematics Subject Classification: 34DXX

Journal of Economic Literature (JEL) Classification : C3, C32

1. INTRODUCTION

Epidemiology is the study of diseases in human population. The nature of spreading the diseases is the need of the hour. In this context mathematical models gain importance in disease prediction analysis. Different mathematical tools are applied in the epidemic analysis. Biological systems are nonlinear in nature. The nonlinear dynamics was studied by L.S.chen and J.chen [11]. R.M. Anderson and May [6] illustrated biological and infectious diseases models. In epidemic models total population is divided into three classes (S, I, R). Let $S(t)$ represent susceptible population, $I(t)$ represent infected population and $R(t)$ represent removable population. Mena-Lorca and Hethcote H.W [5] discussed the SIR epidemic model. We proposed a susceptible infected and removed population with saturated incidence rate for investigation. The saturated incidence rate $\frac{\beta SI}{1 + kS}$ and infective recover saturated function $\frac{rI}{1 + \alpha_1 I}$ are considered. In this model we induced the infective recover saturated function with natural incidence rate. We also included the case where recovered individuals lose the immunity and go back to the susceptible class with the rate θ . The mathematical version is characterized by the couple of non-linear differential equations. Under the investigation the

system has two equilibrium points. Stability analysis is carried out at these points. Lyapunov function is identified to study the global stability analysis and proved that the system is globally asymptotically stable. Numerical simulation is carried out and shown that system is asymptotically stable for suitable parametric values. The infective recover saturated function has a significant impact on stability analysis. Due to this rate, the infected population becomes fever and subsequently susceptible and removable population's increases from its initial level. Due to the saturated incidence rate, periodic oscillations occurs in the three populations makes the system unstable. These two natures are addressed by the set of numerical examples.

2. MATHEMATICAL MODEL

The mathematical model with three compartments (S, I, R) with two saturated incidence rate for the SIR model is given by the following couple of non-linear differential equations

$$\begin{aligned}\frac{dS}{dt} &= b - \frac{\beta SI}{1+kS} - \mu S + \theta R \\ \frac{dI}{dt} &= \frac{\beta SI}{1+kS} - (p + \mu + \alpha)I - \frac{rI}{1+\alpha_1 I} \\ \frac{dR}{dt} &= pI - (\mu + \theta)R + \frac{rI}{1+\alpha_1 I}\end{aligned}\quad (2.1)$$

Where

b = The recruitment rate of the population .

β = The transmission rate.

μ = The natural death rate of the populations.

θ = The rate at which recovered people lose immunity and go back to the susceptible class.

p = The recovery rate of the infective people,

α = The death rate due to disease .

K = The parameter which measures the effect of physiological of other mechanisms

r = Cure rate

α_1 = Non negative constant

The basic reproduction rate of the system (2.1) is given by $R_0 = \frac{b\beta}{\mu(\mu + p + \alpha)}$

If $R_0 > 1$, the disease will spread out and if $R_0 < 1$ there is no disease will spread out

3. STEADY STATE POINTS

The system (2.1) has two steady (Equilibrium) states given by

$$E_1 \text{ Disease free equilibrium point (DFE) is } \left(\frac{b}{\mu}, 0, 0 \right) \quad (3.1)$$

E_2 Endemic equilibrium point (EE) is

$$(\bar{S}, \bar{I}, \bar{R}) = \left(\frac{(p + \mu + \alpha)(1 + \alpha_1 \bar{I}) + r}{\beta(1 + \alpha_1 \bar{I}) - k[(p + \mu + \alpha)(1 + \alpha_1 \bar{I}) + r]}, \frac{b - N\mu}{\mu}, p\bar{I} + \frac{r\bar{I}}{1 + \alpha_1 \bar{I}} \right)$$

where $N = S + I + R$ constant

(3.2)

4. LOCAL STABILITY ANALYSIS

4.1 Local stability at DFE point E_1

Theorem 1: The system is stable at DFE point E_1

Proof: The Jacobean matrix for the system is DFE point (E_1) is given by

$$J = \begin{bmatrix} -\mu & -\frac{\beta b}{(\mu + kb)} & \theta \\ 0 & 0 & 0 \\ 0 & p & -(\mu + \theta) \end{bmatrix}$$

Let the characteristic equation of the system is $\lambda(\lambda^2 + \lambda(2\mu + \theta) + \mu(\mu + \theta)) = 0$ (4.1.1)

Which gives the $\lambda = 0$ and $\lambda^2 + \lambda(2\mu + \theta) + \mu(\mu + \theta) = 0$ (4.1.2)

The equation (7.4.1.1) has negative real roots if the trace of the equation $\left(\frac{-b}{a}\right)$ is negative and the

determinant $\left(\frac{c}{a}\right)$ is positive

Here the trace is $= \frac{-b}{a} = \frac{-(2\mu + \theta)}{1} = -(2\mu + \theta) < 0$

Determinant $= \frac{c}{a} = \frac{\mu(\mu + \theta)}{1} = \mu(\mu + \theta) > 0$

Hence the DEF point $E_1(S^*, I^*, R^*)$ is asymptotically stable.

4.2. Local stability of analysis at endemic equilibrium (EE) point E_2

Theorem 2: The system is locally asymptotically stable at EE point E_2 if $I > S(1 + kS)$

Proof: The variational matrix for the system (2.1) at endemic equilibrium point is given by

$$J = \begin{bmatrix} \frac{-\beta I}{(1 + kS)^2} - \mu & \frac{-\beta S}{(1 + kS)} & \theta \\ \frac{\beta I}{(1 + kS)^2} & \frac{\beta S}{(1 + kS)} - (p + \mu + \alpha) - \frac{r}{(1 + \alpha_1 I)^2} & 0 \\ 0 & p + \frac{r}{(1 + \alpha_1 I)^2} & -(\mu + \theta) \end{bmatrix}$$

With the characteristic equation $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$ (4.2.1)

where

$$a_0 = 1$$

$$a_1 = \left[\frac{r}{(1+\alpha I)^2} + p + 3\mu + \alpha + \theta + \frac{\beta}{(1+kS)^2} (I - S(1+kS)) \right]$$

$$a_2 = \left[\frac{\mu(p+\mu+\alpha) + (\mu+\theta)(p+\mu+\alpha) + \mu(\mu+\theta) + \frac{\beta}{(1+kS)^2} (p+\alpha) + \frac{r\beta I}{(1+kS)^2(1+\alpha I)^2} + \frac{\mu r}{(1+\alpha I)^2} + \frac{\beta(\mu+\theta)}{(1+kS)^2} (I - S(1+kS)) + \frac{\mu\beta}{(1+kS)^2} (I - S(1+kS)) \right]$$

$$a_3 = \left[\frac{\mu(\mu+\theta)(p+\mu+\alpha) + \frac{\alpha(\mu+\theta)\beta I}{(1+kS)^2} + \frac{\beta I \mu p}{(1+kS)^2} + \frac{\mu r \beta I}{(1+kS)^2(1+\alpha I)^2} + \frac{(\mu+\theta)\mu r}{(1+\alpha I)^2} + \frac{\mu(\mu+\theta)\beta}{(1+kS)^2} (I - S(1+kS)) \right]$$

(4.2.2)

Calculate the following determinates $D_1 = a_1$, $D_2 = \begin{vmatrix} a_1 & a_2 \\ a_0 & a_3 \end{vmatrix}$, $D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & 0 & a_3 \end{vmatrix}$

$$D_1 = a_1 = \left[\frac{r}{(1+\alpha I)^2} + p + 3\mu + \alpha + \theta + \frac{\beta}{(1+kS)^2} (I - S(1+kS)) \right] > 0$$

This is a positive value

$$D_2 = \begin{vmatrix} \frac{r}{(1+\alpha I)^2} + p + 3\mu + \alpha + \theta + \frac{\beta}{(1+kS)^2} (I - S(1+kS)) & \frac{\mu(\mu+\theta)(p+\mu+\alpha) + \frac{\alpha(\mu+\theta)\beta I}{(1+kS)^2} + \frac{\beta I \mu p}{(1+kS)^2} + \frac{\mu r \beta I}{(1+kS)^2(1+\alpha I)^2} + \frac{(\mu+\theta)\mu r}{(1+\alpha I)^2} + \frac{\mu(\mu+\theta)\beta}{(1+kS)^2} (I - S(1+kS)) \\ 1 & \frac{\mu(p+\mu+\alpha) + (\mu+\theta)(p+\mu+\alpha) + \mu(\mu+\theta) + \frac{\beta}{(1+kS)^2} (p+\alpha) + \frac{r\beta I}{(1+kS)^2(1+\alpha I)^2} + \frac{\mu r}{(1+\alpha I)^2} + \frac{\beta(\mu+\theta)}{(1+kS)^2} (I - S(1+kS)) + \frac{\mu\beta}{(1+kS)^2} (I - S(1+kS)) \end{vmatrix}$$

$$\begin{aligned}
 &= \left(\frac{r}{(1+\alpha)^2} \left[\frac{\mu(p+\mu\alpha) + (\mu\theta)(p+\mu\alpha) + \mu(\mu\theta)}{(1+kS)^2} + \frac{\beta}{(1+kS)^2} \frac{(p+\alpha)}{(1+\alpha)^2} + \frac{r\beta}{(1+kS)^2(1+\alpha)^2} + \frac{\mu}{(1+\alpha)^2} + \right. \right. \\
 &\quad \left. \left. + \frac{\beta}{(1+kS)^2} \frac{\beta\mu\theta}{(1+kS)^2} (I-S(1+kS)) + \frac{\mu\beta}{(1+kS)^2} (I-S(1+kS)) \right] \right. \\
 &\quad \left. + \frac{\beta}{(1+kS)^2} (I-S(1+kS)) \left[\frac{\mu(p+\mu\alpha) + (\mu\theta)(p+\mu\alpha) + \mu(\mu\theta)}{(1+kS)^2} + \frac{\beta}{(1+kS)^2} \frac{(p+\alpha)}{(1+\alpha)^2} + \frac{r\beta}{(1+kS)^2(1+\alpha)^2} + \frac{\mu}{(1+\alpha)^2} + \right. \right. \\
 &\quad \left. \left. + \frac{\beta}{(1+kS)^2} \frac{\beta\mu\theta}{(1+kS)^2} (I-S(1+kS)) + \frac{\mu\beta}{(1+kS)^2} (I-S(1+kS)) \right] \right. \\
 &\quad \left. + p \left[\frac{\mu(p+\mu\alpha) + (\mu\theta)(p+\mu\alpha) + \mu(\mu\theta)}{(1+kS)^2} + \frac{\beta}{(1+kS)^2} \frac{(p+\alpha)}{(1+\alpha)^2} + \frac{r\beta}{(1+kS)^2(1+\alpha)^2} + \frac{\mu}{(1+\alpha)^2} + \right. \right. \\
 &\quad \left. \left. + \frac{\beta}{(1+kS)^2} \frac{\beta\mu\theta}{(1+kS)^2} (I-S(1+kS)) + \frac{\mu\beta}{(1+kS)^2} (I-S(1+kS)) \right] \right. \\
 &\quad + \frac{2\beta\mu p}{(1+kS)^2} + \frac{2\beta\mu\alpha}{(1+kS)^2} + \frac{2\mu\beta}{(1+kS)^2(1+\alpha)^2} + 3\mu^2(p+\mu\alpha) + \frac{3\mu^2\beta}{(1+kS)^2} (I-S(1+kS)) + \frac{2\beta\mu\theta}{(1+kS)^2} (I-S(1+kS)) + \frac{\beta p\alpha}{(1+kS)^2} + \frac{\beta\alpha^2}{(1+kS)^2} + \frac{\alpha\beta}{(1+kS)^2(1+\alpha)^2} \\
 &\quad + \mu\alpha(p+\mu\alpha) + \frac{2\mu\alpha}{(1+\alpha)^2} + 2\alpha(\mu\theta)(p+\mu\alpha) + \mu\alpha(\mu\theta) + \frac{\alpha\beta}{(1+kS)^2} (I-S(1+kS)) + \frac{(\mu\theta\alpha\beta)}{(1+kS)^2} (I-S(1+kS)) + \frac{\beta p\theta}{(1+kS)^2} + \frac{\beta\alpha\alpha}{(1+kS)^2} + \frac{\theta\beta}{(1+kS)^2(1+\alpha)^2} \\
 &\quad + \mu\theta(p+\mu\alpha) + 2\alpha(\mu\theta)(p+\mu\alpha) + \mu\alpha(\mu\theta) + \frac{\mu\theta}{(1+\alpha)^2} + \frac{\beta\theta}{(1+kS)^2} (I-S(1+kS)) + \frac{\beta\mu\theta}{(1+kS)^2} (I-S(1+kS)) \geq 0
 \end{aligned}$$

D2 is also a positive value $I > S(1+kS)$

$$D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & 0 & a_3 \end{vmatrix}_3 = (a_1 a_2 - a_3) a_3$$

$$\begin{aligned}
 &= \left(\frac{r}{(1+\alpha l)^2} \right) \left[\frac{\mu(p+\mu\alpha) + (\mu\theta)(p+\mu\alpha) + \mu(\mu\theta) + \frac{\beta}{(1+kS)^2}(p+\alpha) + \frac{r\beta l}{(1+kS)^2(1+\alpha l)^2} + \frac{\mu r}{(1+\alpha l)^2} + \right. \\
 &\quad \left. + \frac{\beta}{(1+kS)^2} \frac{\beta(\mu\theta)}{(1+kS)^2(I-S(1+kS)) + \frac{\mu\beta}{(1+kS)^2}(I-S(1+kS))} \right] \\
 &\quad + \frac{\beta}{(1+kS)^2} \left[\frac{\mu(p+\mu\alpha) + (\mu\theta)(p+\mu\alpha) + \mu(\mu\theta) + \frac{\beta}{(1+kS)^2}(p+\alpha) + \frac{r\beta l}{(1+kS)^2(1+\alpha l)^2} + \frac{\mu r}{(1+\alpha l)^2} + \right. \\
 &\quad \left. + \frac{\beta}{(1+kS)^2} \frac{\beta(\mu\theta)}{(1+kS)^2(I-S(1+kS)) + \frac{\mu\beta}{(1+kS)^2}(I-S(1+kS))} \right] \\
 &\quad + p \left[\frac{\mu(p+\mu\alpha) + (\mu\theta)(p+\mu\alpha) + \mu(\mu\theta) + \frac{\beta}{(1+kS)^2}(p+\alpha) + \frac{r\beta l}{(1+kS)^2(1+\alpha l)^2} + \frac{\mu r}{(1+\alpha l)^2} + \right. \\
 &\quad \left. + \frac{\beta}{(1+kS)^2} \frac{\beta(\mu\theta)}{(1+kS)^2(I-S(1+kS)) + \frac{\mu\beta}{(1+kS)^2}(I-S(1+kS))} \right] \\
 &\quad + \frac{2\beta l \mu p}{(1+kS)^2} + \frac{2\beta l \mu \alpha}{(1+kS)^2} + \frac{2\mu r \beta l}{(1+kS)^2(1+\alpha l)^2} + 3\mu^2(p+\mu\alpha) + \frac{3\mu^2\beta}{(1+kS)^2}(I-S(1+kS)) + \frac{2\mu\beta(\mu\theta)}{(1+kS)^2}(I-S(1+kS)) \\
 &\quad + \frac{\beta l p \alpha}{(1+kS)^2} + \frac{\beta l \alpha^2}{(1+kS)^2} + \frac{\alpha r \beta l}{(1+kS)^2(1+\alpha l)^2} \\
 &\quad + \mu \alpha(p+\mu\alpha) + \frac{2\mu \alpha}{(1+\alpha l)^2} + 2\alpha(\mu\theta)(p+\mu\alpha) + \mu \alpha(\mu\theta) + \frac{\alpha \mu \beta}{(1+kS)^2}(I-S(1+kS)) + \frac{(\mu\theta)\alpha\beta}{(1+kS)^2}(I-S(1+kS)) \\
 &\quad + \frac{\beta p \theta}{(1+kS)^2} + \frac{\beta l \theta \alpha}{(1+kS)^2} + \frac{\theta r \beta l}{(1+kS)^2(1+\alpha l)^2} \\
 &\quad + \mu \theta(p+\mu\alpha) + 2\theta(\mu\theta)(p+\mu\alpha) + \mu \theta(\mu\theta) + \frac{\mu \theta}{(1+\alpha l)^2} + \frac{\beta \theta}{(1+kS)^2}(I-S(1+kS)) + \frac{\beta \theta(\mu\theta)}{(1+kS)^2}(I-S(1+kS)) \\
 &\quad + \left[\frac{\mu(\mu\theta)(p+\mu\alpha) + \frac{\alpha(\mu\theta)\beta l}{(1+kS)^2} + \frac{\beta l \mu p}{(1+kS)^2} + \frac{\mu r \beta l}{(1+kS)^2(1+\alpha l)^2} + \frac{(\mu\theta)\mu r}{(1+\alpha l)^2} + \right. \\
 &\quad \left. + \frac{\mu(\mu\theta)\beta}{(1+kS)^2}(I-S(1+kS)) \right] > 0
 \end{aligned}$$

D_3 is also a positive value if $I > S(1+kS)$

Using Routh –Hurwitz criteria all the three determinates are positive. Hence the the system is asymptotically stable if $I > S(1+kS)$

Therefore ,the system is locally asymptotically stable at EE point $(\bar{S}, \bar{I}, \bar{R})$

5. GLOBAL STABILITY

Theorem 3: The system (2.1) is globally asymptotically stable at EE point.

Proof: Consider the Lyapunov's function

$$V(t) = \left[S - \bar{S} - \bar{S} \ln \left(\frac{S}{\bar{S}} \right) \right] + \left[I - \bar{I} - \bar{I} \ln \left(\frac{I}{\bar{I}} \right) \right] + \left[R - \bar{R} - \bar{R} \ln \left(\frac{R}{\bar{R}} \right) \right] \quad (5.1)$$

Differentiate (5.B.4.1) along the solutions (5.B.3.1.1) we get

$$\begin{aligned}
 \dot{V}(t) &= \left[1 - \frac{\bar{S}}{S}\right] \frac{dS}{dt} + \left[1 - \frac{\bar{I}}{I}\right] \frac{dI}{dt} + \left[1 - \frac{\bar{R}}{R}\right] \frac{dR}{dt} \\
 \dot{V}(t) &= \frac{(S - \bar{S})}{S} \left(b - \frac{\beta SI}{1 + kS} - \mu S + \theta R\right) + \frac{(I - \bar{I})}{I} \left(\frac{\beta SI}{1 + kS} - (p + \mu + \alpha)I - \frac{rI}{1 + \alpha_1 I}\right) + \\
 &\quad \frac{(R - \bar{R})}{R} \left(pI - (\mu + \theta)R + \frac{rI}{1 + \alpha_1 I}\right)
 \end{aligned} \tag{5.2}$$

By using the relation from system (2.1) we have

$$\begin{aligned}
 b - \frac{\beta SI}{1 + kS} + \theta R &= \mu \bar{S} \\
 \frac{\beta SI}{1 + kS} - \frac{rI}{1 + \alpha_1 I} &= (p + \mu + \alpha) \bar{I} \\
 pI + \frac{rI}{1 + \alpha_1 I} &= (\mu + \theta) \bar{R}
 \end{aligned} \tag{5.3}$$

From the equation (5.3) and (5.2)

$$\begin{aligned}
 \dot{V}(t) &= \frac{(S - \bar{S})}{S} (\mu \bar{S} - \mu S) + \frac{(I - \bar{I})}{I} ((p + \mu + \alpha) \bar{I} - (p + \mu + \alpha) I) + \\
 &\quad \frac{(R - \bar{R})}{R} ((\mu + \theta) \bar{R} - (\mu + \theta) R) \\
 \dot{V}(t) &= \frac{(S - \bar{S})}{S} \mu (\bar{S} - S) + \frac{(I - \bar{I})}{I} (p + \mu + \alpha) (\bar{I} - I) + \frac{(R - \bar{R})}{R} (\mu + \theta) (\bar{R} - R) \\
 \dot{V}(t) &= -\frac{\mu}{S} (S - \bar{S})^2 - \frac{(p + \mu + \alpha)}{I} (I - \bar{I})^2 - \frac{(\mu + \theta)}{R} (R - \bar{R})^2 \\
 \dot{V}(t) &= -\left[\frac{\mu}{S} (S - \bar{S})^2 + \frac{(p + \mu + \alpha)}{I} (I - \bar{I})^2 + \frac{(\mu + \theta)}{R} (R - \bar{R})^2 \right] \\
 \dot{V}(t) &< 0
 \end{aligned}$$

Hence, the system is globally asymptotically stable at EE point.

6. NUMERICAL SIMULATION

In this section, we carry out numerical simulation using MATLAB for different set of parametric values and draw the graphs. Figure (A) represents Time series responses and (B) represents Phase portraits. consider the following examples for numerical simulation

Example 6.1: Let $S = 13, I = 9, R = 7$

$b = 10.01, \beta = 0.47, k = 0.102, \mu = 0.0964, \theta = 0.91, p = 1.01, \alpha = 1.0068, r = 5.2, \alpha_1 = 3.26301;$

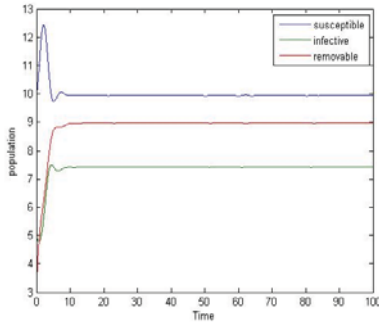


Fig 6.1(A)

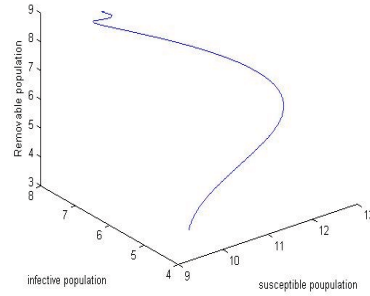


Fig 6.1(B)

For the values in example 1, the system is asymptotically stable at the fixed equilibrium point $E(10, 7, 9)$

Example 6.2: Let $S = 3, I = 2, R = 1$

$b = 19, \beta = 0.47, k = 0.0102, \mu = 0.064, \theta = 0.91, p = 1.01, \alpha = 10.0068, r = 10, \alpha_1 = 3.26301$;

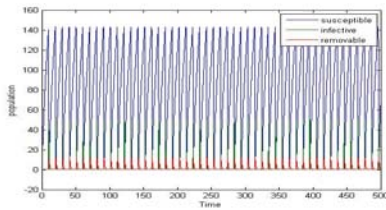


Fig 6.2(A)

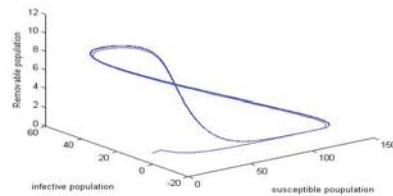


Fig 6.2(B)

For the values in example2, the system exhibits bounded periodic oscillations makes the system unstable.

Example 6.3: Let $S = 3, I = 2, R = 1$

$b = 19, \beta = 0.47, k = 0.0102, \mu = 0.064, \theta = 0.91, p = 1.01, \alpha = 10.0068, r = 22, \alpha_1 = 3.26301$;

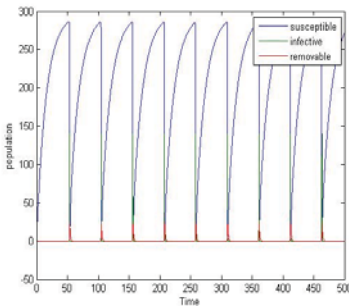


Fig 6.3(A)

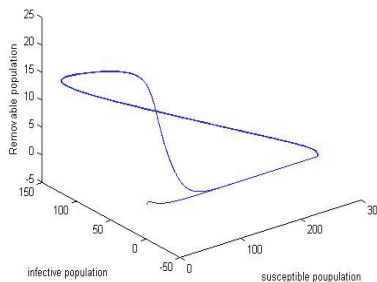


Fig 6.3(B)

For the values in example3, infected and removable populations extinct. A periodic unbounded oscillation in susceptible population makes the system unstable.

Example 6.4: Let $S = 3, I = 2, R = 1$

$b = 19, \beta = 0.47, k = 0.0102, \mu = 0.064, \theta = 0.91, p = 1.01, \alpha = 10.0068, r = 3, \alpha_1 = 3.26301$;

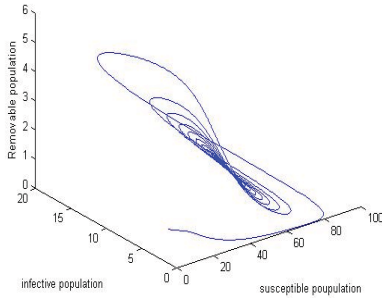


Fig 6.4(A)

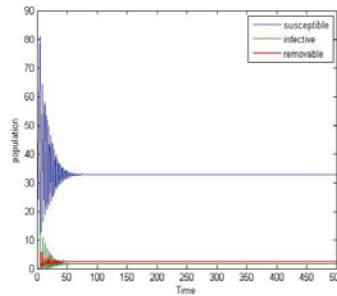


Fig 6.4 (B)

For the values in example 4, the system exhibits bounded oscillations and makes the system asymptotically at the equilibrium point $E[33, 2, 3]$

Example 6.5: Let $S = 10, I = 5, R = 3$

$b = 14, \beta = 0.47, k = 0.0102, \mu = 0.064, \theta = 0.01, p = 1.01, \alpha = 10.0068, r = 3.26301, \alpha_1 = 2$;

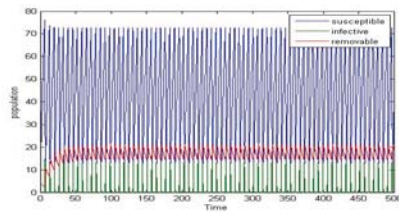


Fig 6.5(A)

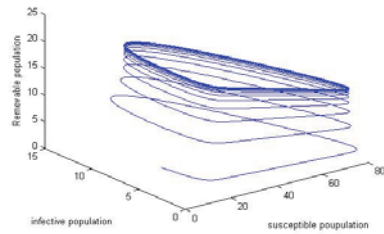


Fig 6.5(B)

For the values in example 5, the system exhibits unstable periodic oscillations in three populations

Example 6.6: Let $S = 10, I = 5, R = 3$

$b = 14, \beta = 0.47, k = 0.0102, \mu = 0.064, \theta = 0.01, p = 1.01, \alpha = 10.0068, r = 3.26301, \alpha_1 = 29$;

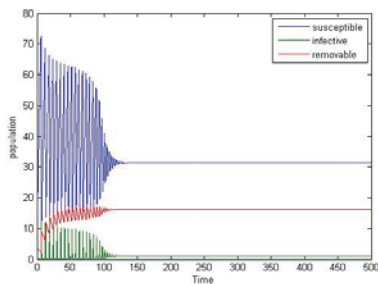


Fig 6.6(A)

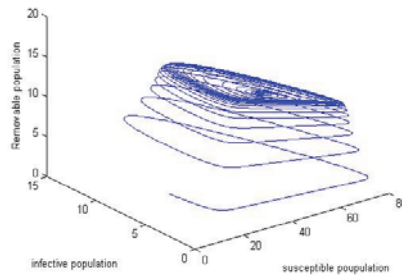


Fig 6.6 (B)

For the values in example 6, the system is stable at Equilibrium point $E[31, 1, 16]$

Example 6.7: Let $S = 10, I = 5, R = 3$

$b = 14, \beta = 0.47, k = 0.0102, \mu = 0.064, \theta = 0.01, p = 1.01, \alpha = 10.0068, r = 1, \alpha_1 = 3.26301$;

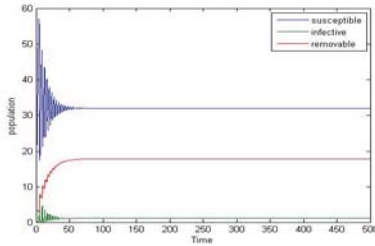


Fig 6.7(A)

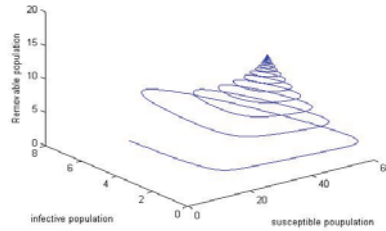


Fig 6.7(B)

For the values in example 7, the system exhibits stable oscillations and moves towards the equilibrium point $E[32, 1, 18]$

Example 6.8: Let $S = 10, I = 5, R = 3$

$b = 14, \beta = 0.47, k = 0.0102, \mu = 0.064, \theta = 0.01, p = 1.01, \alpha = 10.0068, r = 5.2, \alpha_1 = 3.2631$;

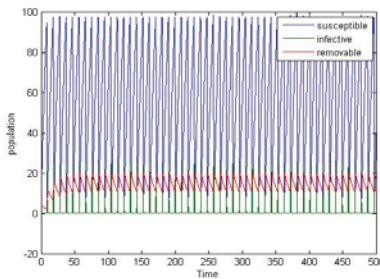


Fig 6.8(A)

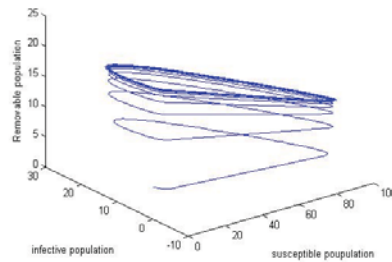


Fig 6.8(B)

For the values in example 8, the system exhibits unstable periodic oscillations, cause system unstable.

Example 6.9: Let $S = 15, I = 10, R = 5$

$b = 17, \beta = 0.37, k = 0.0209, \mu = 0.01264, \theta = 0.0993, p = 8.091, \alpha = 4.9068, r = 0.5, \alpha_1 = 3.26301$;

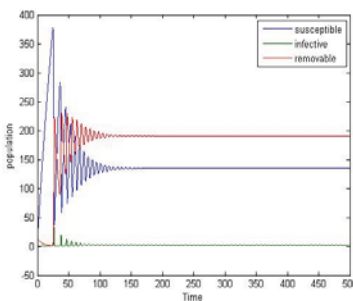


Fig 6.9(A)

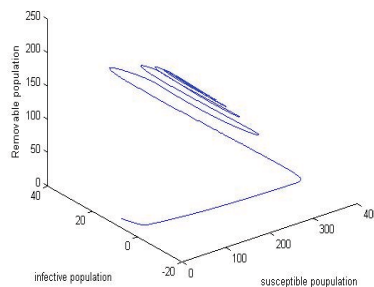


Fig 6.9(B)

For the values in example 9, the system is asymptotically stable at the equilibrium point $E[135, 3, 191]$

Example 6.10: Let $S = 15$, $I = 10$, $R = 5$

$b = 17, \beta = 0.37, k = 0.0209, \mu = 0.01264, \theta = 0.0993, p = 8.091, \alpha = 4.9068,$
 $r = 1, \alpha_1 = 3.26301;$

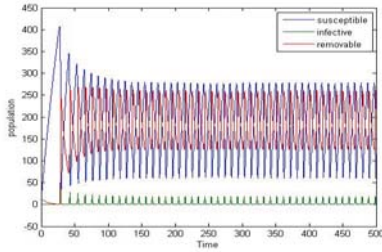


Fig 6.10(A)

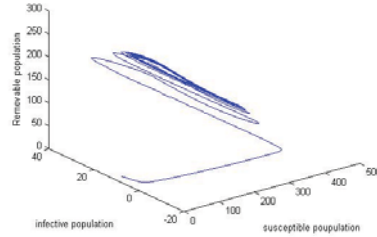


Fig 6.10(B)

For the values in example 10, the system exhibits stable oscillations cause the system unstable. Periodic unbounded oscillations occur in susceptible and removable population.

Example 6.11: Let $S = 15$, $I = 10$, $R = 5$

$b = 17, \beta = 0.37, k = 0.0209, \mu = 0.01264, \theta = 0.0993, p = 8.091, \alpha = 4.9068,$
 $r = 1.3, \alpha_1 = 3.26301;$

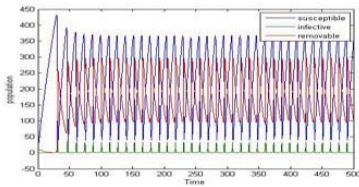


Fig 6.11(A)

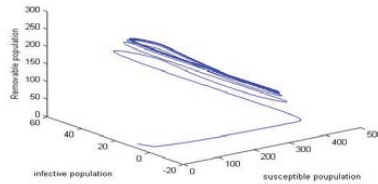


Fig 6.11(B)

For the values in example 11, infected population wash out, susceptible and removable populations rise from its initial population strengths with unbounded periodic oscillations. This makes the system unstable.

Example 6.12: Let $S = 15$, $I = 10$, $R = 5$

$b = 17, \beta = 0.37, k = 0.0209, \mu = 0.01264, \theta = 0.0993, p = 8.091, \alpha = 4.9068,$
 $r = 2, \alpha_1 = 3.26301;$

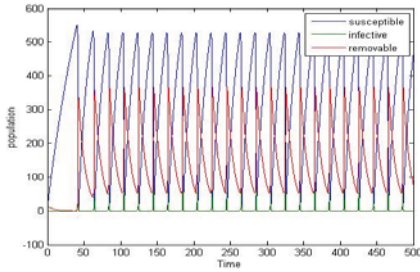


Fig 6.12(A)

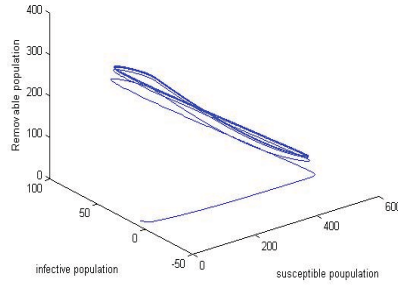


Fig 6.12 (B)

For the values example 12 , the system is unstable due to unbounded oscillations in susceptible and removable population. Infected population extinct

Example 6.13: Let $S = 15$, $I = 10$, $R = 5$

$$b = 14, \beta = 0.37, k = 0.0209, \mu = 0.0264, \theta = 0.91, p = 2.91, \alpha = 2.0068,$$

$$r = 3.26301, \alpha_1 = 0.91;$$

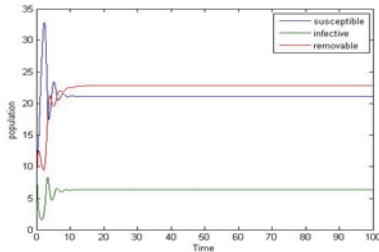


Fig 6.13(A)

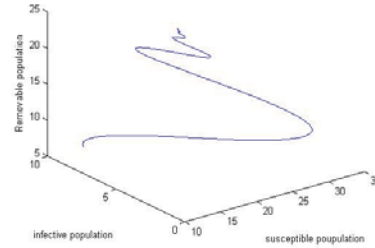


Fig 6.13(B)

For the values in example 13, the system asymptotically stable at the equilibrium point $E[21, 6, 23]$

Example 6.14: Let $S = 15$, $I = 10$, $R = 5$

$$b = 18, \beta = 1.30127, k = 0.02091, \mu = 0.0264, \theta = 0.91, p = 4.91, \alpha = 16.0068, r = 10, \alpha_1 = 3.2631;$$

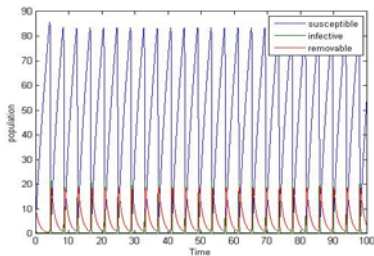


Fig 6.14(A)

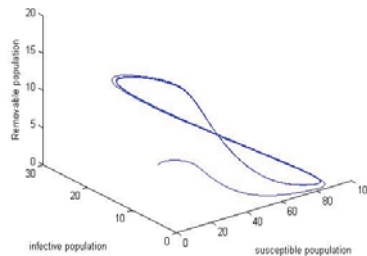


Fig 6.14(B)

For the values in example 14, the system exhibits periodic unbounded oscillations that makes the system unstable. There is no spread of infection.

Example 6.15: Let $S = 15$, $I = 10$, $R = 5$

$b=18, \beta=1.30127, k=0.02091, \mu=0.0264, \theta=0.91, p=4.91, \alpha=16.0068,$
 $r=5, \alpha_1=3.2631;$

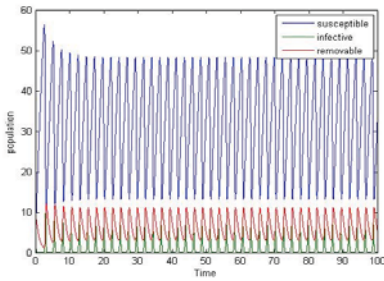


Fig 6.15(A)

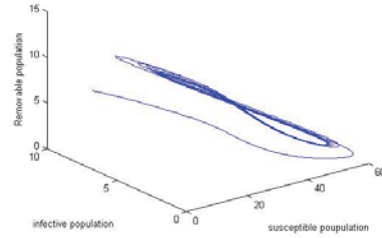


Fig 6.15(B)

For the values in example 15, the system exhibits unstable oscillations and hence the system is unstable

Example 6.16: Let $S = 15, I = 10, R = 5$

$b=18, \beta=1.30127, k=0.02091, \mu=0.0264, \theta=0.91, p=4.91, \alpha=16.0068,$
 $r=4, \alpha_1=3.2631;$

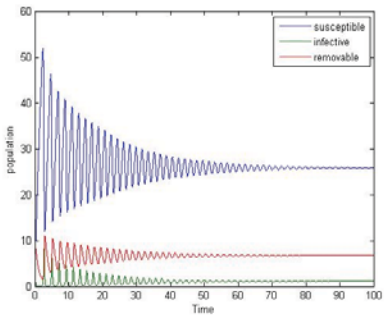


Fig 6.16(A)

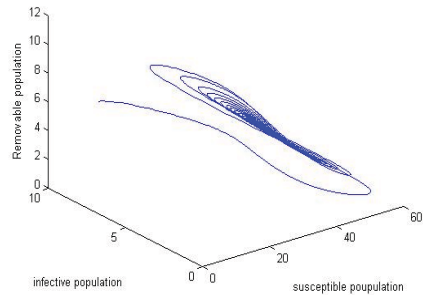


Fig 6.16(B)

For the values in example 16, the system is asymptotically stable at the equilibrium point

7. RESULTS AND CONCLUSION

The chapter deals with the stability analysis of SIR model with saturated incidence rate in susceptible populations and infected population. The model under investigation has two equilibrium points. Disease equilibrium point is asymptotically stable. The endemic equilibrium point is asymptotically stable if $I > S(1 + kS)$. The system is globally asymptotically stable is shown by choosing suitable lyapunov's function. Numerical simulation is carried out in support of stability analysis.

The simulation is carried out for 16 set of examples. The system exhibits periodic unbounded oscillations makes unstable system is shown in the examples 6. 2,3,5,8,10,11,12,14,15. In these examples infected population is almost extinct, Due to the saturated incidence rate included in the model for infected population, the growth rate in removable and susceptible populations are raise

from its population strengths makes un stable system shown in the above examples. The system is asymptotically shown in the examples 6.1, 4, 6, 7, 9, 13,16. Here also low growth in infected population, but no unbounded oscillations are found. Hence incidence rate of infected population makes a significant growth in susceptible and removable populations. Growth rate in susceptible population is high due to saturated incidence in susceptible and removable populations. This factor also influence in the growth rate of removable population.

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