

Dynamics of cannibalism in the predator in a prey refuge functional response dependent predator-prey model

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ABSTRACT

The mathematical model for predator-prey interactions discussed in this article includes cannibalism in the predator. Cannibalism is pervasive in natural individuals which reflects the potential to modify their functional relationship for interacting with resources. We have considered here prey refuge, which depends on the contacts between both the species. Predator consumes the prey according to Holling type II functional response. Feasibility as well as stability of the points of equilibria and the existence of Hopf bifurcation along with its direction at interior equilibrium have been duly accomplished and in this regard suitable graphical representation with proper discussions are executed to substantiate the system under consideration.

Keywords: Predator-prey, Refuge, Cannibalism, Persistence, Stability, Bi-stability, Bifurcation, Numerical Simulation.

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1 Introduction

Due to the complexity of biological processes in both mathematics and ecology, mathematical biology is one of the most essential fields for researchers. Single-species models are adequate to laboratory studies, but in the real world, can deliberate a telescoping of consequences which instigate the population dynamics. Alfred James Lotka [cf. Lotka (1956)] and Vito Volterra [cf. Volterra (1931)] considered two populations and analysed the model. Lotka-Volterra's model for prey-predator is widely considered as one of the basic population model in the dynamical system. After them, Holling [cf. Holling (1959, 1965)] has introduced some

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more realistic predator-prey model considering some type of functional responses for different species with the phenomena of predation. Consumption rate of the prey by its predator bears an important role in the dynamical behaviour of a prey-predator system [cf. Haque et al. (2013); Singh and Sharma (2022)]. Except the predation, there are also some biological factors like migration, refuge, cannibalism etc., that can change the dynamical behavior of a model system. One of the most important environmental phenomena by which a prey species protects itself from predation is refuge. Detritivorous fish and their carnivorous fish predators coexist in the Sundarban, a mangrove area in the delta formed by the confluence of the Padma, Brahmaputra, and Meghna Rivers in the Bay of Bengal. The accumulation of detritus is abundant throughout the environment. The densely flooded bushy parts of the forest provide the detritivorous fish here with simple protection. Predators and prey cohabit in this area in enormous densities. But note that there is also a portion of the forest that has been deliberately reclaimed. There are fewer bushes available where the ecosystem is under a bit of anthropogenic stress. Consequently, there is limited cover for prey. As a result of the easy accessibility to their prey, the predators engage in more competition for the food [cf. Haque et al. (2014)]. Rats, for instance, can sustain a higher population density if they have places to hide from predators like owls and cats, such as tall grass. During the last two decades, many scholars [cf. Sih (1987); Islam and Sarwardi (2023); Nazmul et al. (2021); Sarwardi et al. (2013); Molla et al. (2018); Rahman et al. (2023); Zhou et al. (2019)] extensively investigated the dynamical behaviors on the prey-predator system incorporating prey refuge and they have showed refuge plays key role to determine some interesting behaviour of the system. In the study [cf. Molla et al. (2018)] Molla et al. have already studied the model system allowing prey refuge dependent on the contacts between prey and predator involving the Holling type-II functional response. They have only investigated the role of the prey refuge on the dynamical behaviour of the model system but not considered cannibalism that is omnipresent in the communities. Consuming one's own species' organs or flesh is known as cannibalism. It is a tendency that certain species exhibit in which they ingest other members of their own species in an attempt to capture food [cf. Deng et al. (2019), Walters et al. (2016), Kang et al. (2015)]. Though it is a common ecological interaction in the animal kingdom and is prevalent in aquatic ecosystems, it has also been observed in human beings. While cannibalism diminishes potential competition through regulating the population, it increases the risk of pathogen transmission and decreases expected survival rate of the whole population. Many scholars [cf. Zhang et al. (2019); Basheer et al. (2016, 2018); Chen et al. (2019); Lin et al. (2020); Lin et al. (2020)] investigated the dynamical behaviours of the model system with cannibalism in the species. Cannibalism is common in both prey and predator species. To the best of our knowledge, cannibalism has mostly been considered as a phenomenon in prey species. Basheer et al. [cf. Basheer et al. (2016)] investigated how cannibalism in prey affects the dynamics of the Holling-Tanner type predator-prey model. They found that cannibalism in the prey could weaken rather than stabilise the stable interior equilibrium. Lin et al. [cf. Lin et al. (2020)] analysed the Leslie-Gower predator-prey model with a Holling type-II response function that took cannibalism in the prey into account. They did not consider prey refuge and their study showed that cannibalism did not affect

on the local stability property of the system. Our work is different from them in the sense that it takes into account nonlinear prey refuge and incorporates cannibalism in predator populations. The motivation behind our study lies in the fact that prey hides in a variety of shelters that depend on their interaction with the predator species, and the predator's capability to cannibalise weaker predator species provides them with a second source of food [cf. Abdulghafour and Naji (2022)].

Holling type I, commonly known as the mass action type response function, is one of the most well-known predator-prey response functions. It takes the form mx , where m represents the consumption rate and x represents the volume of prey population. If the captured prey is not handled, this model behaves well. This assumption is widely considered to be too obtuse [cf. Korpimäki and Norrdah (1991)]. In reality, predators frequently feel full, which causes their predation rate to decline in the presence of abundant food supplies. It is obvious that a linear function cannot depict this trait. When the populations are not very large, the Holling type I response function is generally mathematically convenient. To better understand and to make more realistic of the consumption rate, the Holling type II functional response $\frac{mx}{a+x}$ is used in our model system. It is a hyperbola with horizontal asymptote, that deliberates the maximum predation rate. Based on Molla et al. [Molla et al. (2018)] and Deng et al. [Deng et al. (2019)] we have proposed our model system and our present model is a modification of Molla et al. [Molla et al. (2018)] through the incorporation of cannibalism in predator.

In the investigation, we have set up our remaining works as follows: In Section 2, we have proposed the basic assumptions and then construct the model equations. This Section also includes boundedness of the model system. The Section 3 deals with the existence of feasible equilibrium. The stability behaviour of the system around the points of equilibrium are performed in Section 4. Further, this Section contains the persistence of model system. Bifurcation analyses of the dynamical system are dealt with the Section 5. The analytical findings are validated via numerical simulations in Section 6, and the conclusion part of the present investigation is carried out in Section 7.

2 The basic model

This Section relates with the formulation of model system. In order to develop the system, the following presumptions have been considered:

- (i) A logistic type growth function of prey species,
- (ii) Holling type II functional response,
- (iii) Nonlinear prey refuge is considered because, in some natural system, prey refuge may be impacted by both the volume of predators and prey, and
- (iv) Amount of cannibalism in predator population.

Let any time $t > 0$, x and y be the number of prey and predator population (per unit area) respectively. Based on the above assumptions, we formulate the following system:

$$\begin{aligned}\frac{dx}{dt} &= x(a - bx) - \frac{m(x - \theta xy)y}{a_1 + (x - \theta xy)}, \\ \frac{dy}{dt} &= y(c_1 - \beta) + \frac{em(x - \theta xy)y}{a_1 + (x - \theta xy)} - \frac{cy^2}{y + d},\end{aligned}\quad (2.1)$$

where $c_1 < c$, a and b denote the intrinsic growth rate and intra-specific competition of the prey population respectively, a_1 is the half saturation constant, β is the death rate of the predator, e is the conversion efficiency of ingested prey into new predators, m denotes the per capita consumption rate, θ is the coefficient of prey refuge. Assuming the amount of prey refuge θxy [cf. Molla et al. (2018), Haque and Sarwardi (2018)], predators get recourse to the remaining prey species $(x - \theta xy)$. The cannibalism of predators is termed by $\frac{cy^2}{y+d}$, in which c signifies the cannibalism rate, c_1 is the birth rate caused by cannibalism in predators and d is the half-saturation constant of cannibalism. It is assumed that $0 < \theta < 1$, and $0 \leq (1 - \theta y) \leq 1$ throughout the manuscript. We take only those values of θ such as $(1 - \theta y) \geq 0$, i.e., $y \leq \frac{1}{\theta}$, which assures the acceptable range of refuge, i.e., $0 \leq (1 - \theta y) \leq 1$. Here the parameters $a, b, c_1, m, \theta, a_1, \beta, e, c, d$ of the proposed system are all positive. We investigate the system (2.1) in the region $\Gamma = \{(x, y) \in \mathbb{R}_+^2 : x \geq 0, y \geq 0\}$. The aim of this paper is to uncover the dynamical behaviors of the predator-prey model (2.1) and discuss the effects of cannibalism on the system.

2.1 Boundedness of the system

Boundedness assures the consistency of a biological system. Using standard precise arguments, one can easily find that the solution of the system(2.1) always exists and is positive.

Theorem 2.1. *In the system (2.1), both the prey and predator population are bounded from the above.*

Proof. The first equation of the system (2.1) gives

$$\frac{dx}{dt} = x(a - bx) - \frac{mx(1 - \theta y)y}{a_1 + x(1 - \theta y)} < x(a - bx) = bx\left(\frac{a}{b} - x\right),$$

therefore, by the simple standard arguments we have, $\lim_{t \rightarrow +\infty} \sup x(t) < \frac{a}{b}$.

Also second equation of the system (2.1) assures that

$$\begin{aligned}\frac{dy}{dt} &= y(c_1 - \beta) + \frac{emxy(1 - \theta y)}{a_1 + x(1 - \theta y)} - \frac{cy^2}{y + d}, \\ &< c_1 y + emy - \frac{cy^2}{y + d} \\ &< (c_1 + em)y - \frac{cy^2}{d} \\ &= \frac{cy}{d} \left(\frac{d(c_1 + em)}{c} - y \right),\end{aligned}$$

hence, by a standard arguments we get, $\lim_{t \rightarrow +\infty} \sup y(t) < \frac{(c_1 + em)d}{c}$.

Hence the system is dissipative. Now there exists a $T > 0$ such that for all $t > T$, we have $x(t) < \frac{a}{b} + \epsilon =: M_1$ and $y(t) < \frac{c_1 a_1 b + aem}{aem\theta} + \epsilon =: M_2$, where ϵ is a very small positive number. $\square$

3 Existence and local stability analyses of the points of equilibria:

We are keen to investigate the dynamical behaviour of the system in the first quadrant, $R_0 \times R_0 = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$. A system is at equilibrium when neither the population level is changing, i.e., the derivatives $\dot{x}$ and $\dot{y}$ are simultaneously zero. The non-negative equilibrium points of the system (2.1) are as follows:

- (i) The trivial equilibrium point $E_0(0, 0)$.
- (ii) Predator free equilibrium point $E_1(\frac{a}{b}, 0)$, which is unconditionally feasible.
- (iii) Prey free equilibrium point $E_2(0, \frac{d(c_1 - \beta)}{\beta + c - c_1})$, which is feasible if $\beta < c_1 < \beta + c$.
- (iv) Interior equilibrium point $E_3(x_2, y_2)$, where y_2 is a positive root of

$$\sum_{i=0}^{i=5} B_i z^i = 0, \quad (3.1)$$

where B_i ($i=0$ to 5) are given in **Appendix 1**. Due to complication of finding all the five real feasible roots, we only find out the sufficient condition of exactly one positive real root, say η and it is discussed in **Appendix 2** and x_2 is a positive root of the following quadratic equation

$$b(1 - \theta y_2)x^2 + (a_1 b - a(1 - \theta y_2))x + m y_2(1 - \theta y_2) - a a_1 = 0, \quad (3.2)$$

The equation (3.2) has atleast one positive real root if $m y_2(1 - \theta y_2) < a a_1$.

Remark 3.1. A realistic ecosystem should not be capable of attaining equilibrium E_0 when there are no organisms present. At this stage, the equilibrium ought to be unstable. In some circumstances, where there is no predator population, it should be feasible to observe equilibrium around E_1 . The system should reach at equilibrium point E_2 , where only the predator population survives if the predators rely on food sources other than their focal prey. The equilibrium point E_3 , where prey and predator cohabit, is quite common in natural ecosystems and should be stable in some environments.

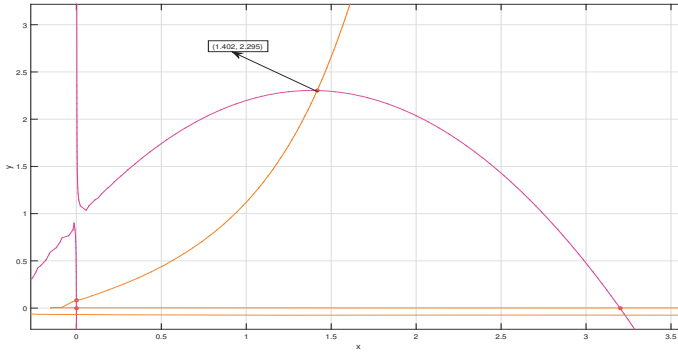


Figure 1: Nullclines of the system (2.1), the parameters values are used from the Table 2 with $c = 4.49$. The red dotted point is an interior equilibrium point of the system.

4 Stability analysis of the system:

This Section discusses about the stability analysis of the system (2.1) around the various equilibrium points. If a system's state is initiated close to, but not exactly at a given equilibrium point, local stability analysis can be used to establish whether or not the system will eventually settle to that point. The qualitative behaviour of a nonlinear system in the neighbourhood of an equilibrium point will be similar to the that of zero equilibrium of the corresponding linear system provided the equilibrium point of the nonlinear system is hyperbolic. The Jacobian matrix $J = (a_{ij})_{2 \times 2}$ of the system at equilibrium point (x, y) is:

$$J = \begin{bmatrix} a - 2bx - \frac{a_1my(1-\theta y)}{[a_1+x(1-\theta y)]^2} & \frac{mx(2a_1y\theta - \theta^2y^2x + 2xy\theta - a_1 - x)}{[a_1+x(1-\theta y)]^2} \\ \frac{a_1emy(1-\theta y)}{[a_1+x(1-\theta y)]^2} & c_1 - \beta + \frac{emx(1-2\theta y)}{a_1+x(1-\theta y)} + \frac{emx^2y\theta(1-\theta y)}{[a_1+x(1-\theta y)]^2} - \frac{2cy}{y+d} + \frac{cy^2}{(y+d)^2} \end{bmatrix}$$

4.1 Analysis of Local stability

- (i) The system (2.1) is unstable around the trivial equilibrium point E_0 .
- (ii) The equilibrium point E_1 is asymptotically stable if $\beta > c_1 + \frac{ema}{a+a_1b}$.
- (iii) The equilibrium point E_2 , if exists, is stable if $aa_1 < \frac{mdk_1(c_1-\beta)}{\beta+c-c_1}$ where $k_1 > 0$ and $k_1 = 1 - \frac{\theta d(c_1-\beta)}{\beta+c-c_1}$. Hence E_2 is locally asymptotically stable in the region $\Lambda_1 = \{(c, \theta) \in \mathbb{R}_+^2 : \theta < \theta_1\}$, where $\theta_1 = \frac{md(c_1-\beta)(\beta+c-c_1)-aa_1(\beta+c-c_1)^2}{md^2(c_1-\beta)^2}$ (see Figure 2).
- (iv) Let the jacobian matrix at the coexistence equilibrium point $E_3(x_2, y_2)$, if exists, be $(\xi_{ij})_{2 \times 2}$ $i, j = 1, 2$; where ξ_{ij} are given in **Appendix 3**. Next we see, $\text{tr} J_3 = \xi_{11} + \xi_{22} < 0$ for $m < \frac{b(a_1+x_2-\theta x_2 y_2)^2}{y_2(1-\theta y_2)^2}$ and $\det J_3 = \xi_{11}\xi_{22} - \xi_{12}\xi_{21} > 0$ in the region $\Lambda_2 = \{(c, \theta) \in \mathbb{R}_+^2 : \theta < \theta_2\}$, where θ_2 is given in the **Appendix 4**.

Proposition 4.1. *The system has bistability region $\Lambda_3 = \{(c, \theta) \in \mathbb{R}_+^2 : \theta_1 < \theta < \theta_2\}$. The region is shown in Figure 2.*

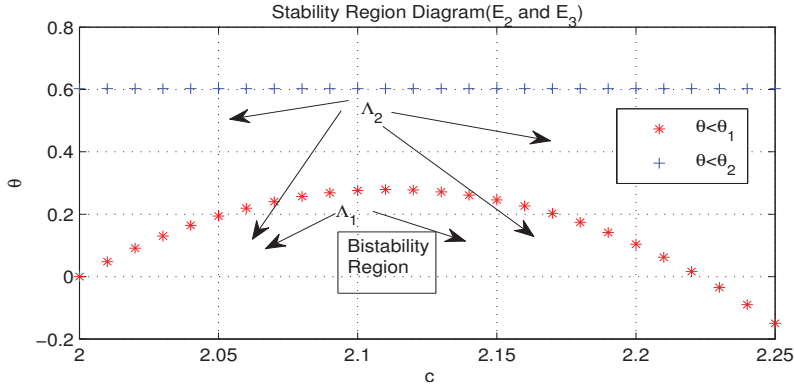


Figure 2: Numerical investigation of the region in $c - \theta$ parameter space of locally asymptotic stability of E_2 and E_3 . The other parameters values are taken from the Table 2.

Theorem 4.2. *The system (2.1) shows global stability behaviour around prey free equilibrium point $E_2(0, \frac{d(c_1-\beta)}{\beta+c-c_1})$ if both the conditions $c_1 < \beta + c$ and $aa_1b < \frac{bmd(c_1-\beta)}{\beta+c-c_1} + \frac{ac_1a_1b+a^2em}{em}$, are satisfied simultaneously.*

Proof. To prove the theorem, we will construct a suitable a Lyapunov function L_1 [cf. Rahman et al. (2018)] given by $L_1 = x + \frac{1}{e}(y - y_1 - y_1 \ln \frac{y}{y_1})$, where $y_1 = \frac{d(c_1-\beta)}{\beta+c-c_1}$.

Taking the time derivative, we obtain

$$\begin{aligned} \frac{dL_1}{dt} &= x(a - bx) - \frac{m(x - \theta xy)y}{a_1 + (x - \theta xy)} + \frac{1}{e}(y - y_1)(c_1 - \beta + \frac{em(x - \theta xy)}{a_1 + (x - \theta xy)} - \frac{cy}{y + d}) \\ &= ax - bx^2 - \frac{my_1(x - \theta xy)}{a_1 + (x - \theta xy)} - \frac{cd}{e}(y - y_1)^2 \\ &\leq 0, \text{ if } a - \frac{my_1(1 - \theta y)}{a_1 + (x - \theta xy)} < 0, \end{aligned}$$

which provides that the system is globally asymptotically stable around prey free equilibrium point $E_2(0, \frac{d(c_1-\beta)}{\beta+c-c_1})$ if $aa_1b < \frac{bmd(c_1-\beta)}{\beta+c-c_1} + \frac{ac_1a_1b+a^2em}{em}$. $\square$

Theorem 4.3. *The system (2.1) shows global stability behaviour around $E_3 = (x_2, y_2)$ if the followings sufficient conditions are satisfied:*

- (i) $\frac{b}{m} > \frac{(1-\theta y_2)y_2}{a_1^2}$ and
- (ii) $B^2 - 4AC < 0$, where $A = \frac{b}{m} - \frac{(1-\theta y_2)y_2}{a_1^2}$, $B = -\frac{\theta y_2}{a_1}$, $C = \frac{cd}{e(H+d)(y_2+d)} + \frac{mba_1\theta x_2}{(a+a_1b)(a_1+x_2)}$ and $H = \frac{(c_1+em)d}{c}$.

Proof. Let us consider a scalar function $L_2: \mathbb{R}^2 \rightarrow \mathbb{R}$, given by

$$L_2 = \{x - x_2 - x_2 \ln \frac{x}{x_2}\} + \frac{1}{e}\{y - y_2 - y_2 \ln \frac{y}{y_2}\}.$$

$$\begin{aligned}
 \frac{dL_2}{dt} &= (x - x_2) \frac{\dot{x}}{x} + \frac{1}{e} (y - y_2) \frac{\dot{y}}{y} \\
 &= [a - bx - \frac{m(1 - \theta y)y}{a_1 + (x - \theta xy)}] (x - x_2) + \frac{1}{e} [c_1 - \beta + \frac{(1 - \theta y)}{a_1 + (x - \theta xy)} - \frac{cy}{y + d}] (y - y_2) \\
 &\leq -m(\frac{b}{m} - \frac{(1 - \theta y)(1 - \theta y_2)y_2}{(a_1 + x(1 - \theta y)(a_1 + x_2(1 - \theta y_2)))}) (x - x_2)^2 + \frac{m\theta y_2}{a_1} (x - x_2)(y - y_2) \\
 &\quad - m(\frac{a_1\theta x_2}{(a_1 + x(1 - \theta y))(a_1 + x_2(1 - \theta y_2))} + \frac{cd}{e(y + d)(y_2 + d)})(y - y_2)^2, \\
 &\leq -[A(x - x_2)^2 + B(x - x_2)(y - y_2) + C(y - y_2)^2],
 \end{aligned}$$

therefore, $\frac{dL_2}{dt} < 0$, along all the trajectories in the first quadrant except the point $E_3 = (x_2, y_2)$ if the condition $B^2 - 4AC < 0$ is satisfied. Again $\frac{dL_2}{dt} = 0$ at $E_3 = (x_2, y_2)$. Hence it follows that L_2 is a Lyapunov function and from Lyapunov-Lasalle's invariance principle [cf. Hale (1989)], it follows the result. $\square$

4.2 Persistence

Biologically, persistence of a system means the survival of all the species, that is if all the species are initially coexisting, then they will continue to coexist.

Theorem 4.4. *The system (2.1) is persistent under the conditions: (i) $c_1 > \beta + \frac{cy_1}{y_1 + d}$ and (ii) $aa_1 > my_1(1 - \theta y_1)$, where $y_1 = \frac{d(c_1 - \beta)}{\beta + c - c_1}$.*

Proof. To prove the theorem, we have formed average Lyapunov function [cf. Gard and Hallam (1979), Freedman et al. (1984)]. Let us consider a function $V(x, y) = x^h y^k$, where $h, k > 0$.

Now, we have

$$\begin{aligned}
 \Xi(x, y) &= \frac{\dot{V}(x, y)}{V(x, y)} \\
 &= h \frac{\dot{x}}{x} + k \frac{\dot{y}}{y} \\
 &= h[(a - bx) - \frac{m(1 - \theta y)y}{a_1 + x(1 - \theta y)}] + k[c_1 - \beta + \frac{emx(1 - \theta y)}{a_1 + x(1 - \theta y)} - \frac{cy}{y + d}].
 \end{aligned}$$

Then,

$\Xi(0, 0) = ha + k(c_1 - \beta) > 0$ if $c_1 > \beta$, whereas,

$\Xi(\frac{a}{b}, 0) = k(c_1 - \beta + \frac{ema}{a + a_1 b}) > 0$ if $c_1 > \beta$, and

$\Xi(0, \frac{d(c_1 - \beta)}{\beta + c - c_1}) = h[a - \frac{my_1(1 - \theta y_1)}{a_1}] + k[c_1 - \beta - \frac{cy_1}{y_1 + d}] > 0$, if $aa_1 > my_1(1 - \theta y_1)$ and $c_1 > \beta + \frac{cy_1}{y_1 + d}$.

So the values of Ξ at all boundary points are positive under the conditions (i) and (ii). Thus V is an average Lyapunov function and hence the proof. $\square$

5 Bifurcation analyses of system (2.1)

Bifurcation of a dynamical system may occur when parameter value of a system changes. As a parameter crosses a critical value, the stability behaviour of the equilibrium point changes and a bifurcation occurs. In the following, we will discuss about different types of bifurcation.

5.1 Hopf-bifurcation for the dynamical system :

When a system exhibits Hopf-bifurcation with respect to a parameter, it signifies that the parameter has a critical value from which the stability of the system switches and a periodic solution develops from a stable state. Hopf bifurcation assures that all populations can exist in an oscillatory manner.

Here, we analyze the existence of Hopf-bifurcation at the interior equilibrium point $E_3(x_2, y_2)$ assuming c as Hopf-bifurcation parameter. The Characteristic equation of the Jacobian matrix at E_3 is of the form

$$\tau^2 + S\tau + T = 0, \quad (5.1)$$

where, $S = -\text{tr}J_3$ and $T = \det J_3$. Let us write the eigenvalue of the Jacobian matrix at E_3 in a generic form as $\tau(c) = \alpha(c) + i\gamma(c)$. Putting this value in the equation (5.1) and comparing the real and imaginary parts, we get

$$\alpha^2 - \gamma^2 + S\alpha + T = 0, \quad 2\alpha\gamma + S\gamma = 0. \quad (5.2)$$

If the equation (5.1) has a pair of purely imaginary eigenvalue at the critical value of $c = c_c$, i.e., $\alpha(c_c) = 0$ and $\gamma(c_c) \neq 0$, then the equilibrium point losses its stability. Next, from (5.2), we have $-\gamma^2(c_c) + T(c_c) = 0$, $S(c_c) = 0$.

Thus we get the necessary condition for Hopf-bifurcation that $S(c_c) = 0$ and $\gamma = \sqrt{T}(c_c)$ and it is found the transversality condition for Hopf-bifurcation, $\frac{d\alpha}{dc} = -\frac{1}{2} \frac{y_2(c)d}{(y_2(c)+d)^2} \neq 0$ at $c = c_c$. This ensures the existence of Hopf-bifurcation.

5.2 Direction of Hopf-bifurcation

From the Subsection (5.1), we have showed the conditions under which the system experiences a Hopf-bifurcation near the interior equilibrium point $E_3(x_2, y_2)$ at the critical value of $c = c_c$. The following quantities are obtained from **Appendix 6** by applying the centre manifold theorem credited to [cf. Hassard et al. (1981)].

$$\begin{aligned} C_1(0) &= \frac{i}{2\gamma_0} (g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{C_1(0)}{\text{Re}(\tau'(0))}, \\ \beta_2 &= 2\text{Re}(C_1(0)), \\ \tau_2 &= -\frac{\text{Im}(C_1(0)) + \mu_2\text{Im}(\tau'(0))}{\gamma_0}, \end{aligned}$$

where g_{ij} are given in **Appendix 6**. Here the parameters μ_2, β_2, τ_2 determine the direction of Hopf-bifurcation, stability of bifurcating periodic solutions and the period of bifurcating solutions respectively. The nature of Hopf-bifurcating periodic solutions at $c = c_c$ are determined as follows:

- (i) The Hopf-bifurcation is subcritical or supercritical according as $\mu_2 < 0$ or > 0 .

- (ii) The periodic solutions are stable or unstable in nature if the value of $\beta_2 < 0$ or > 0 .
- (iii) The period of the periodic solution decreases or increases according as the value of $\tau_2 < 0$ or > 0 .

5.3 Analyses of other bifurcation

A bifurcation at some point occurs in a dynamical system when a parameter is smoothly varied but the behaviour of the system in the neighbourhood of this point suddenly changes. Here, we have discussed whether the system possesses some other kind of bifurcation except Hopf-bifurcation.

One of the eigenvalues of the Jacobian matrix J_3 at E_3 will be zero if and only if $\det J_3 = \xi_{11}\xi_{22} - \xi_{21}\xi_{12} = 0$, which gives $c = c^{[sn]}$. The other eigenvalue is $\text{tr} J_3 = \xi_{11} + \xi_{22}$, evaluated at $c = c^{[sn]}$. Next let Ψ and Υ be the eigenvectors corresponding to eigenvalue 0 of the matrix J_3 and its transpose respectively. We obtain that $\Psi = (\varpi_1, \varpi_2)^T$, $\Upsilon = (h_1, h_2)^T$, where $\varpi_1 = -\frac{\xi_{12}\varpi_2}{\xi_{11}} = -\frac{\xi_{22}\varpi_2}{\xi_{21}}$, $h_1 = -\frac{\xi_{21}h_2}{\xi_{11}} = -\frac{\xi_{22}h_2}{\xi_{12}}$ and ϖ_2, h_2 are any two non-zero real numbers. Now $\Upsilon^T[F_c(E_3, c^{[sn]})] = -\frac{h_2 y_2^2}{y_2 + d} \neq 0$ (here, $F_c = \frac{\partial F}{\partial c}$) and $\Upsilon^T[D^2 F(E_3, c^{[sn]})(\Psi\Psi)] \neq 0$. As a result, the system exhibits a saddle-node bifurcation at $c = c^{[sn]}$ around the equilibrium point E_3 [cf. Sotomayor (1973)].

6 Numerical Simulation

Without numerical justification, theoretical analysis can never be fulfilled. To verify the theoretical outcomes of the study, we carried out all numerical simulations of the derived results in this Section. The sufficient conditions of local stability around the equilibrium points and the global stability around prey free equilibrium point E_2 and the coexistence equilibrium point $E_3(x_2, y_2)$ are schematically shown in Table 1. Also the default values with respective units of the parameters of the model system are given in Table 2. Using the parameters from the Table 2 with cannibalism rate $c = 4.49$, we have displayed the nullclines of the system (2.1) in Figure 1, the red dotted point is the interior equilibrium points of the system. To validate the numerical findings in Proposition 1, we take $c = 2.23$ and $\theta = 0.054$ together with the other parametric values as in the Table 2. The bistability region of E_2 and E_3 are showed from the Figure 2.

We observe that for the parametric values in Table 2 except $a = 0.95$, the condition $aa_1 = 0.38 < \frac{mdk_1(c_1 - \beta)}{\beta + c - c_1} = 0.3837$ of local stability at the prey free equilibrium point E_2 is satisfied, whereas keeping the same parametric values if we increase c_1 to 2.65, the system shows the global stability behaviour as the condition of global stability given by the **Theorem 4.2** is satisfied. The corresponding phase diagram is shown in the Figure 3. This confirms that the predators can survive on cannibalism apart from their focal prey. Next, taking the parameters values as in Table 2, when $c = 4.49$ the sufficient conditions of local stability at the co-existence equilibrium are successfully satisfied. In this respect, Figure 4 has been depicted. And the system exhibits limit cycle oscillations when we decrease the cannibalism rate to $c_c = 4.46$, it means all the populations survive in oscillatory mode. Figure 5

has been executed in this regard. Bifurcation diagrams are also displayed in the Figure 6 and this diagram indicates that the periodic oscillation can be suppressed by increasing the cannibalism rate c . For finding the direction of Hopf-bifurcation, we have calculated $\gamma_0 = \sqrt{\det J_3} = 1.716169836$. Also we evaluate from **Appendix 6**, $g_{11} = -1.431334556 - 5.5813004025i$, $g_{02} = -1.373285161 - 5.5021567625i$, $g_{20} = -1.562062824 + 0.6337665188i$, $g_{21} = -0.0132810862 - 0.0456633036i$. Then we get, $C_1(0) = -2.287797121 - 20.85847090i$. Also we evaluate $\alpha'(0) = -0.037389784$, $\gamma'(0) = -0.006120482562$, $\mu_2 = -61.18775976$, $\beta_2 = -4.575594242$, $\tau_2 = 11.96412551$. Since $\beta_2 < 0$, the bifurcating periodic solutions emitting from $E_3 = (1.425, 2.3072)$ are stable in nature. The period of the periodic solution is 11.96412551. The positive sign of τ_2 indicates the increasing period of the periodic solution of the system. Also $\mu_2 < 0$ implies the Hopf-bifurcation of the system occurring at $c_c = 4.46$ is subcritical that is also evident from the Figure 6. Taking $\theta = 0.029$, $a = 11.2$, $b = 6.5$, $e = 0.6$, $a_1 = 2.78$, $c_1 = 2.5$, $d = 0.1$, $\beta = 0.5$, $c = 4.05$, $m = 4.5$, we evaluate that $\frac{b}{m} = 1.444444444 > \frac{(1-\theta y_2)y_2}{a_1^2} = 0.8831990576$, $A = 0.5612453864$, $B = -0.9778633094e - 1$, $C = 0.3131399570$, and $B^2 - 4AC = -0.6934312582 < 0$. Thus the sufficient conditions of global stability given in the **Theorem 4.3** are successfully satisfied. The global stability occurs at the point $(0.04171, 9.374)$ and the corresponding phase portrait is shown in the Figure 7. Another important behaviour of the system (2.1) is the change the of stability from prey free equilibrium point E_2 to the coexistence equilibrium point E_3 . Taking $a = 0.95$, $b = 3.5$, $\theta = 0.029$, $e = 0.6$, $a_1 = 0.4$, $d = 0.1$, $\beta = 0.5$, $m = 5$, $c_1 = 2.5$, we observe that for $c < c^{[tn]} = 4.6258$, the system (2.1) is stable around the prey free equilibrium point E_2 but the coexistence equilibrium point E_3 does not exist, whereas for $c > c^{[tn]}$ the coexistence equilibrium point E_3 become stable and the prey free equilibrium point E_2 becomes unstable. The corresponding phase diagram is shown in the Figure 8. The effects of cannibalism on both the species have been observed. It is seen that gradual increment of cannibalism rate diminishes the predator population and increases the prey population. That means cannibalism among predator population prolong the prey population in the ecosystem (see Figure 9).

Table 1: Schematic representation(LAS-Locally Asymptotically Stable, GAS-Globally Asymptotically Stable, HB-Hopf Bifurcation, SNB-Saddle Node Bifurcation)

Equilibria	Conditions for stability	Equilibrium Nature
E_0	No conditions	Unstable
E_1	$\beta > c_1 + \frac{ema}{a+a_1b}$	LAS
E_2	$\theta < \theta_1$	LAS
E_2	$c_1 > \beta + \frac{aa_1c}{md+aa_1}$	GAS
E_3	$\theta < \theta_2$	LAS
E_3	$\frac{b}{m} > \frac{y_2(1-\theta y_2)}{a_1^2}$ and $B^2 - 4AC < 0$	GAS
E_3	$c = c_c$	HB
E_3	$c = c^{[sn]}$	SNB

Table 2: Biological meanings, default values with respective units of the parameters. Here, time is denoted by T and species volume is denoted by V .

Parameter	Description	Unit	Value
a	Intrinsic growth rate of prey	T^{-1}	11.2
b	Intra-specific competition of prey	$V^{-1}T^{-1}$	3.5
θ	Prey refuge constant	V^{-1}	0.029
m	Consumption rate of predator	$V^{-1}T^{-1}$	5
a_1	Half saturation constant	V	0.4
e	Conversion factor	Unitless	0.6
c_1	Birth rate from predator cannibalism	T^{-1}	2.5
β	Death rate of the predator	T^{-1}	0.5
c	Cannibalism rate	$V^{-1}T^{-1}$	-
d	The half-saturation constant of cannibalism	V	0.1

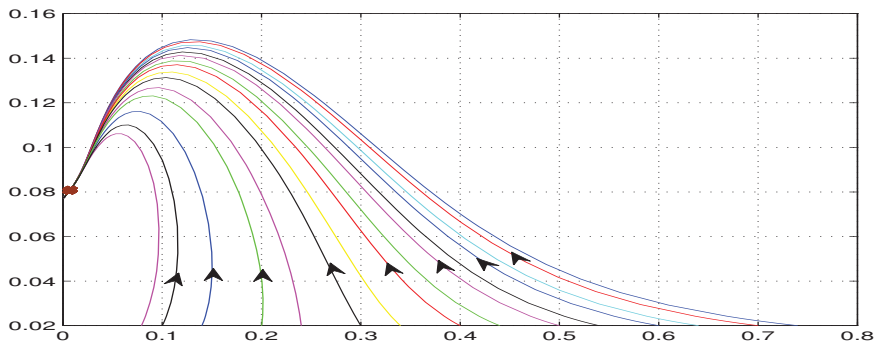
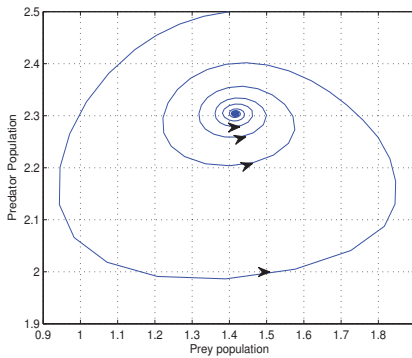
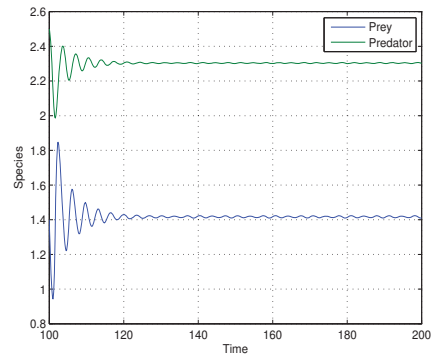


Figure 3: Global stability behaviour of the system (2.1) around the prey free equilibrium point (where horizontal and vertical axis denotes the prey and predator population respectively). Here, the parameters are same as Table 2 except $a = 0.95$ and $c_1 = 2.65$.



(a)



(b)

Figure 4: Local stability around the interior equilibrium point for the parametric value $c = 4.49$ and the other parametric values are taken from the Table 2. (a) Phase diagram, (b) Time series solution.

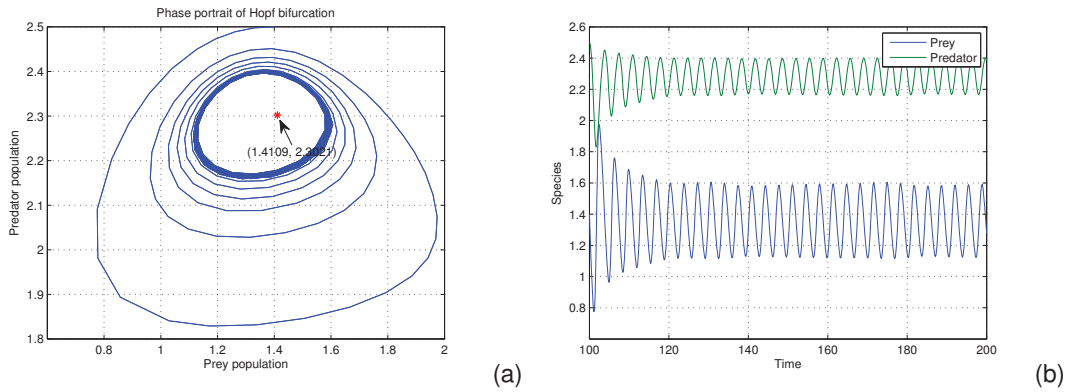


Figure 5: Hopf-bifurcation around the interior equilibrium point of the model system (2.1) for parametric value $c = c_c = 4.46$ and the other parametric values are the same as Table 2. (a) Phase diagram, (b) Time series solution.

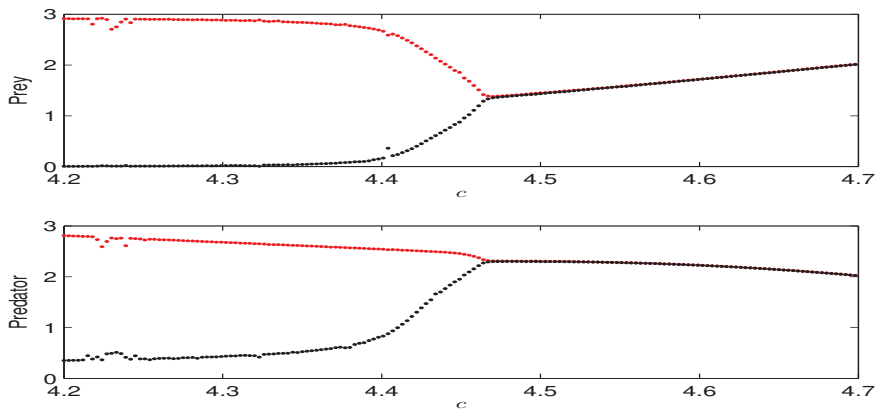


Figure 6: Bifurcation diagram around the interior equilibrium point with respect to the bifurcating parameter c , whereas, the other parameters values are taken from Table 2.

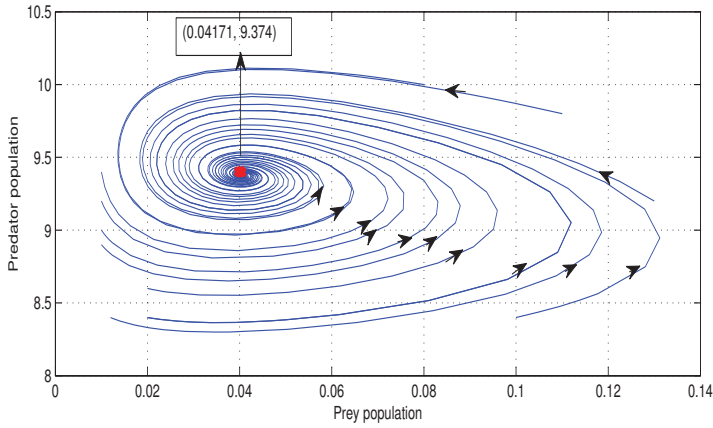


Figure 7: Phase portrait of global stability of the system (2.1), taking the value $a = 11.2$, $b = 6.5$, $\theta = 0.029$, $e = 0.6$, $a_1 = 2.78$, $c_1 = 2.5$, $d = 0.1$, $\beta = 0.5$, $c = 4.05$, $m = 4.5$ satisfying $\frac{b}{m} = 1.44 > \frac{(1-\theta y_2)y_2}{a_1^2} = 0.8832$ and $A = 0.56124$, $B = -0.0978$, $C = 0.004812$, $B^2 - 4AC = -0.010803 < 0$.

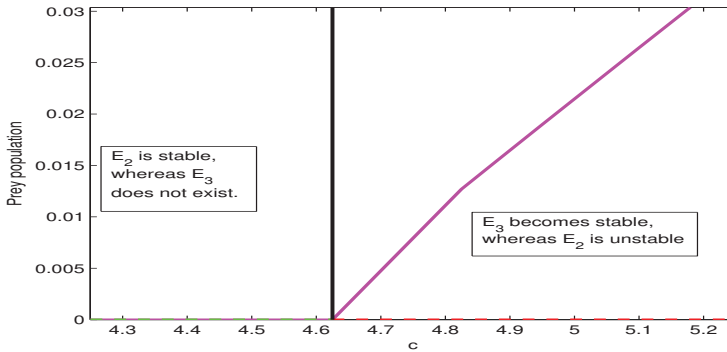
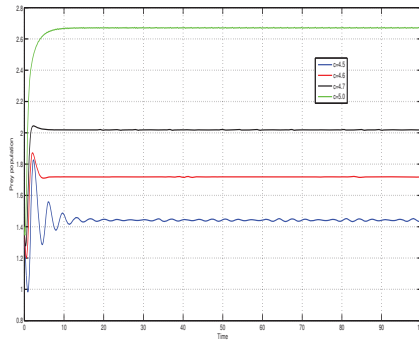
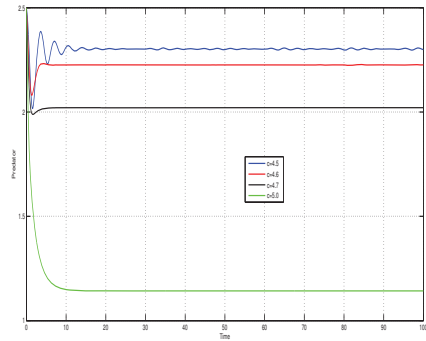


Figure 8: Stability changes from E_2 to E_3 , where green dotted line represents the range of c for the stability of E_2 and non-existence of E_3 , whereas red dotted lines represents the range of c for the stability of E_3 and instability of E_2 . Here the other parametric values are considered as $a = 0.95$, $b = 3.5$, $\theta = 0.029$, $e = 0.6$, $a_1 = 0.4$, $d = 0.1$, $\beta = 0.5$, $m = 5$, $c_1 = 2.5$.



(a)



(b)

Figure 9: (a) Effects of cannibalism rate(c) on prey population, (b) Effect of cannibalism rate(c) on predator population. The other parametric values are the same as in Table 2.

Proposition 6.1. *In the interval $[4,5]$, there exists a critical value c_c , where a subcritical Hopf-bifurcation occurs. For subcritical Hopf-bifurcation in the interval $[4,5]$, the interior equilibrium point is asymptotically stable between c_c and 4.5 and for c lying between 4.45 and c_c , a limit cycle occurs.*

7 Conclusion

This article investigates the dynamics of a prey-predator model with cannibalism in predator population. We have observed that the cannibalism parameter c bears the key role to determine the nature of points of equilibrium. It is evident from the bifurcation diagram that an unstable state of coexistence equilibrium transforms into a stable position when the cannibalism rate c rises. The parameter c has an important role in stability change from the equilibrium point E_2 to E_3 . In Molla et al. [cf. Molla et al. (2018)], prey free equilibrium does not exist but in our system, it is to be noted that the prey-free equilibrium point exists under the condition $\beta < c_1 < \beta + c$, and this equilibrium point is stable under some conditions, is also numerically justified. This indicates that even in the absence of the prey population, the predator population may still survive through cannibalism. For instance, we can say about salamanders that they depend solely on cannibalism to survive during the summer. And this important results are obtained due to modification of the study [cf. Molla et al. (2018)] by introducing cannibalism in predator population. Deng et al. [cf. Deng et al. (2019)] did not address the existence or direction of the Hopf bifurcation in their study. But, in our investigation, we have used proper numerical simulation to analyse and validate the theoretical study of stability and various types of bifurcation. The results show that the cannibalism parameter c is an important factor to determine the dynamical nature of the model system. Deng et al. [cf. Deng et al. (2019)] used Holling type-I response function incorporating cannibalism in predator, whereas our article is more updated in the sense that our model system incorporates Holling type-II response function with the phenomenon of cannibalism in predator population. This makes our model system much more complicated and realistic. The previous authors [cf. Zhang et al. (2019); Basheer

et al. (2016, 2018)] did not find bistability behaviour in their work, whereas our model system possesses bistable region in $c - \theta$ space. Certainly, the dynamical nature of the population is greatly affected by cannibalism.

Our study exhibits very interesting dynamical behaviors. The study can be modified by considering other types of functional response. The system can also be refined by incorporating some biological factors like fear effect, harvesting policy of the population and wind effect. In the near future, we are interested to expand our work to apply these schemes.

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8 Appendices

8.1 Appendix 1

$$\begin{aligned}
 B_0 &= -ba_1^2ed^2c_1 + ba_1^2ed^2\beta - a_1aed^2c_1 + a_1aed^2\beta - aa_1e^2md^2, \\
 B_1 &= ba_1^2edc + a_1aedc + d^2\beta^2 - 2d^2em\beta + e^2m^2d^2 - 2a_1ae^2md - 2a_1aec_1d + aa_1e^2md^2\theta \\
 &\quad + 2ba_1^2ed\beta - 2c_1d^2\beta + c_1^2d^2 + 2c_1d^2em + 2a_1aed\beta - a_1a\theta ed^2\beta + a_a\theta ed^2c_1 - 2ba_1^2ec_1d, \\
 B_2 &= 2\beta^2d - 4m\beta ed - 2m^2e^2d^2\theta + 4m\beta d^2e\theta - 2\beta^2d^2\theta + 2m^2e^2d - 4mc_1d^2e\theta - a_1ae^2m \\
 &\quad + 4mec_1d + 4c_1d^2\beta\theta - a_1aec_1 + a_1ae\beta - ba_1^2ec_1 + ba_1^2e\beta + 2a_1aec_1d\theta - 2a_1a\theta e\beta d \\
 &\quad + 2a_1ae^2md\theta - 4\beta c_1d - 2c_1^2d^2\theta + 2c_1^2d + 2c\beta d - 2mced - a_1a\theta edc - 2cc_1d + a_1aec \\
 &\quad + ba_1^2ec, \\
 B_3 &= m^2e^2 - 4m^2e^2d\theta + m^2\theta^2e^2d^2 + \beta^2 + 8m\beta ed\theta - 2m\theta^2\beta d^2e - 4\beta^2d\theta + \theta^2\beta^2d^2 - 2m\beta e\theta \\
 &\quad + aa_1e^2m - 8mec_1d\theta + 2m\theta^2c_1d^2e + a_1a\theta ec_1 - a_1a\theta e\beta + 8\beta c_1d\theta - 2\theta^2c_1d^2\beta + c_1^2 - 4c_1^2d\theta \\
 &\quad + \theta^2c_1^2d^2 + 2mec_1 - 2\beta c_1 - 2mec + 2\beta c - 4c\beta d\theta + 4mced\theta + c^2 - 2c_1c - a_1a\theta ec \\
 &\quad + 4cc_1d\theta, \\
 B_4 &= -4\theta^2\beta c_1d + 4c_1c\theta + 2\theta^2\beta^2d + 4m\beta e\theta + 2m^2\theta^2e^2d - 2c_1^2\theta - 4mec_1\theta + 4mec\theta \\
 &\quad + 4m\theta^2ec_1d - 2m\theta^2ced - 2\theta^2cc_1d + 2\theta^2c\beta d - 2m^2e^2\theta + 4\beta c_1\theta - 2\beta^2\theta - 4m\theta^2\beta ed \\
 &\quad + 2\theta^2c_1^2d - 2c^2\theta - 4c\beta\theta, \\
 B_5 &= -2m\theta^2\beta e + 2m\theta^2ec_1 + 2\theta^2\beta c - 2\theta^2c_1c + \theta^2c_1^2 + \theta^2c^2 + m^2\theta^2e^2 + \theta^2\beta^2 - 2m\theta^2ec \\
 &\quad - 2\theta^2\beta c_1.
 \end{aligned}$$

8.2 Appendix 2

Here, we are interested to get a positive root of the equation (3.1). Suppose there exist two complex root α_1 and β_1 , with their complex conjugate α_1^* and β_1^* respectively and a real root η . So the equation (3.1) has two quadratic factor with negative discriminant and let those factors be $z^2 + l_1z + m_1 = (z - \alpha_1)(z - \alpha_1^*)$, $z^2 + l_2z + m_2 = (z - \beta_1)(z - \beta_1^*)$, where $l_1 = -2Re(\alpha_1)$, $l_2 = -2Re(\beta_1)$, $m_1 = |\alpha_1|^2$, $m_2 = |\beta_1|^2$.

Hence, m_1, m_2 are all positive. Then we can write

$$\sum_{i=0}^5 B_i z^i = B_5 [(z^2 + l_1z + m_1)(z^2 + l_2z + m_2)(z - \eta)],$$

Then comparing the constant term of both sides we get, $-m_1m_2\eta = \frac{B_0}{B_5}$ i.e., $\eta = -\frac{B_0}{B_5m_1m_2}$. Then η will be positive if $\frac{B_0}{B_5} < 0$.

8.3 Appendix 3

$$\begin{aligned}
 \xi_{11} &= -bx_2 + \frac{mx_2y_2(1-\theta y_2)^2}{[a_1+x_2(1-\theta y_2)]^2}; \quad \xi_{12} = -\frac{mx_2(a_1+x_2+\theta^2y_2^2x_2^2-2a_1y_2\theta-2x_2y_2\theta)}{[a_1+x_2(1-\theta y_2)]^2}; \quad \xi_{21} = \frac{a_1emy_2(1-\theta y_2)}{[a_1+x_2(1-\theta y_2)]^2}; \\
 \xi_{22} &= -y_2\left(\frac{a_1emx_2\theta}{[a_1+x_2(1-\theta y_2)]^2} + \frac{cd}{(y_2+d)^2}\right).
 \end{aligned}$$

8.4 Appendix 4

$$\theta_2 = \frac{S_1 - \sqrt{S_2}}{2x_2y_2(x_2^2y_2^2b + 2x_2^2db y_2 + x_2^2bd^2 + cd x_2y_2 - m y_2^3 - 2dm y_2^2 - d^2m y_2)}, \text{ where}$$

$$\begin{aligned} S_1 = & 2y_2^2x_2^2ba_1 + 2x_2^3y_2^2b + 2cdx_2^2y_2 + 4y_2x_2^3db + 4x_2^2y_2dba_1 + 2cda_1x_2y_2 + 2x_2^3d^2b + 2d^2ba_1x_2^2 \\ & - 2mx_2y_2^3 - x_2y_2^2ma_1e - 4x_2y_2^2dm - 2x_2y_2dma_1e - 2x_2y_2d^2m - x_2d^2ma_1e, \\ S_2 = & 4x_2^2y_2^5m^2a_1e + x_2^2y_2^4m^2a_1^2e^2 + x_2^2d^4m^2a_1^2e^2 + 4x_2^2y_2^5mba_1^2 - 16x_2^3y_2^3ba_1^2dme - 24x_2^3y_2^3ba_1^2d^2me \\ & - 4x_2^3y_2^3ma_1ecd - 16x_2^4y_2^3ma_1edb - 4x_2^2y_2^3ma_1^2ecd - 24x_2^4y_2^2ma_1ed^2b - 8y_2^2x_2^3d^2ma_1ec \\ & - 8x_2^2y_2^2d^2ma_1^2ec - 16x_2^4y_2d^3ma_1eb - 16x_2^3y_2d^3ma_1^2eb - 4x_2^3y_2cd^3ma_1e - 4cd^3x_2^2y_2a_1^2me \\ & + 16x_2^2y_2^4m^2da_1e + 24x_2^2y_2^3m^2d^2a_1e - 4y_2^4x_2^3ba_1^2me - 4x_2^4y_2^4ma_1eb + 4x_2^2y_2^3m^2a_1^2e^2d \\ & + 6x_2^2y_2^2m^2a_1^2e^2d^2 + 16x_2^2y_2^2d^3m^2a_1e + 4x_2^2y_2d^3m^2a_1^2e^2 + 4x_2^2y_2d^4m^2a_1e - 4x_2^4d^4ma_1eb \\ & - 4x_2^3d^4ma_1^2eb + 16x_2^2y_2^4mdba_1^2 + 4x_2y_2^4mcd a_1^2 + 24x_2^2y_2^3md^2ba_1^2 + 8x_2y_2^3d^2mca_1^2 \\ & + 16x_2^2y_2^2d^3mba_1^2 + 4x_2y_2^2d^3mca_1^2 + 4x_2^2y_2d^4mba_1^2. \end{aligned}$$

8.5 Appendix 5

For the eigenvector $u = (u_1, u_2)^T$, corresponding to an eigenvalue of the matrix J_3 , we define the following operator:

$$D^2F(x, \theta)(u, u) = \begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} u_1^2 + \frac{\partial^2 F_1}{\partial x \partial y} u_1 u_2 + \frac{\partial^2 F_1}{\partial y \partial x} u_2 u_1 + \frac{\partial^2 F_1}{\partial y^2} u_2^2 \\ \frac{\partial^2 F_2}{\partial x^2} u_1^2 + \frac{\partial^2 F_2}{\partial x \partial y} u_1 u_2 + \frac{\partial^2 F_2}{\partial y \partial x} u_2 u_1 + \frac{\partial^2 F_2}{\partial y^2} u_2^2 \end{pmatrix}$$

8.6 Appendix 6

The following formulas are used to calculate the expressions for g_{11}, g_{02}, g_{21} and g_{20} :

We express the dynamical system (2.1) as $\frac{dX}{dt} = F(X)$, where $X = (x, y)$ and $F = (F_1, F_2)$. By taking $c = c_c + \sigma$, so that $\sigma = 0$ is the Hopf-bifurcation value for the system (2.1). Let us consider $z_1 = x - x_2, z_2 = y - y_2$ to translate the equilibrium point to the origin. Applying Taylor series expansion to the system (2.1), the system is developed as follows:

$$\begin{aligned} \frac{dz_1}{dt} &= \xi_{11}z_1 + \xi_{12}z_2 + \sum_{m+n \geq 2}^{\infty} \frac{\alpha_{mn}}{m!n!} x_1^m x_2^n, \\ \frac{dz_2}{dt} &= \xi_{21}z_1 + \xi_{22}z_2 + \sum_{m+n \geq 2}^{\infty} \frac{\beta_{mn}}{m!n!} x_1^m x_2^n. \end{aligned} \quad (8.1)$$

Where, $m, n \geq 0$ and $\alpha_{mn} = \frac{\partial^{m+n} F_1}{\partial^m x \partial^n y} |_{E_3}$; $\beta_{mn} = \frac{\partial^{m+n} F_2}{\partial^m x \partial^n y} |_{E_3}$ and ξ_{ij} are given in **Appendix 3**. In order to identify the characteristics of the Hopf-bifurcating periodic solution, we now take into account the comparable system for additional analysis. The eigenvector λ for eigenvalue $\alpha + i\gamma$ is $\lambda = \begin{bmatrix} 1 \\ \frac{\xi_{11} - \alpha - i\gamma}{-\xi_{12}} \end{bmatrix}$, where $\alpha = \frac{\xi_{11} + \xi_{22}}{2}$ and $\gamma = \frac{1}{2} \sqrt{4(\xi_{12}\xi_{21} + \xi_{11}\xi_{22}) - (\xi_{11} + \xi_{22})^2}$,

let $M = (Re(\tau), -Im(\tau)) = \begin{bmatrix} 1 & 0 \\ \frac{\xi_{11} - \alpha}{-\xi_{12}} & -\frac{\gamma}{\xi_{12}} \end{bmatrix}$ and $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = M^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, then we get the

following system in terms of w_1, w_2 as follows:

$$\begin{aligned}\frac{dw_1}{dt} &= \alpha w_1 - \gamma w_2 + \phi(w_1, w_2; \theta), \\ \frac{dw_2}{dt} &= \gamma y_1 + \alpha y_2 + \Psi(w_1, w_2; \theta).\end{aligned}\quad (8.2)$$

Where,

$$\begin{aligned}\phi(w_1, w_2; \theta) &= \frac{1}{2}\alpha_{20}w_1^2 + \alpha_{11}w_1\phi(w_1, w_2) + \frac{1}{2}\alpha_{02}\phi^2(w_1, w_2) + \frac{1}{2}\alpha_{21}w_1^2\phi(w_1, w_2) \\ &+ \frac{1}{2}\alpha_{12}w_1\phi^2(w_1, w_2) + \frac{1}{6}\alpha_{30}w_1^3 + \frac{1}{6}\alpha_{03}\phi^3(w_1, w_2) + h.o.t,\end{aligned}$$

$$\begin{aligned}\Psi(w_1, w_2; \theta) &= -\frac{\xi_{12}}{\gamma}\left[\frac{1}{2}\beta_{20}w_1^2 + \beta_{11}w_1\phi(w_1, w_2) + \frac{1}{2}\beta_{02}\phi^2(w_1, w_2) + \frac{1}{2}\beta_{12}w_1\phi^2(w_1, w_2)\right. \\ &+ \left.\frac{1}{2}\beta_{21}w_1^2\phi(w_1, w_2) + \frac{1}{6}\beta_{30}w_1^3 + \frac{1}{6}\beta_{03}\phi^3(w_1, w_2)\right] \\ &+ \frac{-\xi_{11} + \alpha}{\gamma}\left[\frac{1}{2}\alpha_{20}w_1^2 + \alpha_{11}w_1\phi(w_1, w_2) + \frac{1}{2}\alpha_{02}\phi^2(w_1, w_2) + \frac{1}{2}\alpha_{21}w_1^2\phi(w_1, w_2)\right. \\ &+ \left.\frac{1}{2}\alpha_{12}w_1\phi^2(w_1, w_2) + \frac{1}{6}\alpha_{30}w_1^3 + \frac{1}{6}\alpha_{03}\phi^3(w_1, w_2)\right] + h.o.t,\end{aligned}$$

and $\phi(w_1, w_2) = \frac{-\xi_{11} + \alpha}{\xi_{12}}w_1 - \frac{\gamma}{\xi_{12}}w_2$, where h.o.t stands for higher order terms. As the given system satisfy all the conditions of Hopf-bifurcation at $E_3(x_2, y_2)$, then we have $\alpha = 0$, $\gamma = \gamma_0 = \sqrt{\det J_3}$ and the system (8.2) reduces to

$$\begin{aligned}\frac{dw_1}{dt} &= -\gamma_0 w_2 + \phi_0(w_1, w_2; \theta), \\ \frac{dw_2}{dt} &= \gamma_0 w_1 + \Psi_0(w_1, w_2; \theta).\end{aligned}\quad (8.3)$$

Where,

$$\begin{aligned}\phi_0(w_1, w_2; \theta) &= \frac{1}{2}\alpha_{20}w_1^2 + \alpha_{11}w_1\phi_0(w_1, w_2) + \frac{1}{2}\alpha_{02}\phi_0^2(w_1, w_2) \\ &+ \frac{1}{2}\alpha_{21}w_1^2\phi_0(w_1, w_2) + \frac{1}{2}\alpha_{12}w_1\phi_0^2(w_1, w_2) + \frac{1}{6}\alpha_{30}w_1^3 + \frac{1}{6}\alpha_{03}\phi_0^3(w_1, w_2) + h.o.t,\end{aligned}$$

$$\begin{aligned}\Psi_0(w_1, w_2; \theta) &= -\frac{\xi_{12}}{\gamma_0}\left[\frac{1}{2}\beta_{20}w_1^2 + \beta_{11}w_1\phi_0(w_1, w_2) + \frac{1}{2}\beta_{02}\phi_0^2(w_1, w_2) + \frac{1}{2}\beta_{12}w_1\phi_0^2(w_1, w_2)\right. \\ &+ \left.\frac{1}{2}\beta_{21}w_1^2\phi_0(w_1, w_2) + \frac{1}{6}\beta_{30}w_1^3 + \frac{1}{6}\beta_{03}\phi_0^3(w_1, w_2)\right] \\ &+ \frac{-\xi_{11}}{\gamma_0}\left[\frac{1}{2}\alpha_{20}w_1^2 + \alpha_{11}w_1\phi_0(w_1, w_2) + \frac{1}{2}\alpha_{02}\phi_0^2(w_1, w_2) + \frac{1}{2}\alpha_{21}w_1^2\phi_0(w_1, w_2)\right. \\ &+ \left.\frac{1}{2}\alpha_{12}w_1\phi_0^2(w_1, w_2) + \frac{1}{6}\alpha_{30}w_1^3 + \frac{1}{6}\alpha_{03}\phi_0^3(w_1, w_2)\right] + h.o.t,\end{aligned}$$

where, $\phi_0(w_1, w_2) = \eta w_1 + \zeta w_2$, $\eta = -\frac{\xi_{11}}{\xi_{12}}$, $\zeta = -\frac{\gamma_0}{\xi_{12}}$.

Next, we find the values of g_{11} , g_{02} , g_{12} and g_{21} at $c = c_c$ and $(w_1, w_2) = (0, 0)$ in the following:

$$\begin{aligned}
 g_{11} &= \frac{1}{4} \left[\frac{\partial^2 \phi_0}{\partial w_1^2} + \frac{\partial^2 \phi_0}{\partial w_2^2} + i \left(\frac{\partial^2 \Psi_0}{\partial w_1^2} + \frac{\partial^2 \Psi_0}{\partial w_2^2} \right) \right] \\
 &= \frac{1}{4} \left[\alpha_{20} + 2\eta\alpha_{11} + (\eta^2 + \zeta^2)\alpha_{02} + i \left(-\frac{\xi_{12}}{\gamma_0} (\beta_{20} + 2\eta\beta_{11} + (\eta^2 + \zeta^2)\beta_{02}) \right. \right. \\
 &\quad \left. \left. - \frac{\xi_{11}}{\gamma_0} (\alpha_{20} + 2\eta\alpha_{11} + (\eta^2 + \zeta^2)\alpha_{02}) \right) \right], \\
 g_{02} &= \frac{1}{4} \left[\frac{\partial^2 \phi_0}{\partial w_1^2} - \frac{\partial^2 \phi_0}{\partial w_2^2} - 2 \frac{\partial^2 \phi_0}{\partial w_1 \partial w_2} + i \left(\frac{\partial^2 \Psi_0}{\partial w_1^2} - \frac{\partial^2 \Psi_0}{\partial w_2^2} + 2 \frac{\partial^2 \Psi_0}{\partial w_1 \partial w_2} \right) \right] \\
 &= \frac{1}{4} \left[\alpha_{20} + 2\eta\alpha_{11} + (\eta^2 - \zeta^2)\alpha_{02} + 2\zeta \frac{\xi_{12}}{\gamma_0} (\beta_{11} + \eta\beta_{02}) + 2\zeta \frac{\xi_{11}}{\gamma_0} (\alpha_{11} + \eta\alpha_{02}) \right. \\
 &\quad \left. + i \left(-\frac{\xi_{12}}{\gamma_0} (\beta_{20} + 2\eta\beta_{11} + (\eta^2 - \zeta^2)\beta_{02}) - \frac{\xi_{11}}{\gamma_0} (\alpha_{20} + 2\eta\alpha_{11} + (\eta^2 - \zeta^2)\alpha_{02}) + 2\zeta(\alpha_{11} + \eta\alpha_{02}) \right) \right], \\
 g_{20} &= \frac{1}{4} \left[\frac{\partial^2 \phi_0}{\partial w_1^2} - \frac{\partial^2 \phi_0}{\partial w_2^2} + 2 \frac{\partial^2 \phi_0}{\partial w_1 \partial w_2} + i \left(\frac{\partial^2 \Psi_0}{\partial w_1^2} - \frac{\partial^2 \Psi_0}{\partial w_2^2} - 2 \frac{\partial^2 \Psi_0}{\partial w_1 \partial w_2} \right) \right] \\
 &= \frac{1}{4} \left[\alpha_{20} + 2\eta\alpha_{11} + (\eta^2 - \zeta^2)\alpha_{02} - 2\zeta \frac{\xi_{12}}{\gamma_0} (\beta_{11} + \eta\beta_{02}) - 2\zeta \frac{\xi_{11}}{\gamma_0} (\alpha_{11} + \eta\alpha_{02}) \right. \\
 &\quad \left. + i \left(-\frac{\xi_{12}}{\gamma_0} (\beta_{20} + 2\eta\beta_{11} + (\eta^2 - \zeta^2)\alpha_{02}) - \frac{\xi_{11}}{\gamma_0} (\alpha_{20} + 2\eta\alpha_{11} + (\eta^2 - \zeta^2)\alpha_{02}) - 2\zeta(\alpha_{11} + \eta\alpha_{02}) \right) \right], \\
 g_{21} &= \frac{1}{8} \left[\frac{\partial^3 \phi_0}{\partial w_1^3} + \frac{\partial^3 \phi_0}{\partial w_1 \partial w_2^2} + \frac{\partial^3 \Psi_0}{\partial w_1^2 \partial w_2} + \frac{\partial^3 \Psi_0}{\partial w_2^3} + i \left(\frac{\partial^3 \Psi_0}{\partial w_1^3} + \frac{\partial^3 \Psi_0}{\partial w_1 \partial w_2^2} - \frac{\partial^3 \phi_0}{\partial w_1^2 \partial w_2} - \frac{\partial^3 \phi_0}{\partial w_2^3} \right) \right] \\
 &= \frac{1}{8} \left[\alpha_{30} + 3\eta\alpha_{21} + (3\eta^2 + \zeta^2)\alpha_{12} + \eta^2(\eta + \zeta)\alpha_{03} - \zeta \frac{\xi_{12}}{\gamma_0} (\beta_{21} + 2\eta\beta_{12} + (\eta^2 + \zeta^2)\beta_{03}) \right. \\
 &\quad \left. - \zeta \frac{\xi_{11}}{\gamma_0} (\alpha_{21} + 2\eta\alpha_{12} + (\eta^2 + \zeta^2)\alpha_{03}) + i \left(-\frac{\xi_{12}}{\gamma_0} (\beta_{30} + 3\eta\beta_{21} + (3\eta^2 + \zeta^2)\beta_{12} + \eta^2(\eta + \zeta)\beta_{03}) \right. \right. \\
 &\quad \left. \left. - \zeta \frac{\xi_{11}}{\gamma_0} (\alpha_{30} + 3\eta\alpha_{21} + (3\eta^2 + \zeta^2)\alpha_{12}) + \eta^2(\eta + \zeta)\alpha_{03} - \zeta(\alpha_{21} + 2\eta\alpha_{12} + (\eta^2 + \zeta^2)\alpha_{03}) \right) \right].
 \end{aligned}$$

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